

# A CONNECTEDNESS THEOREM OVER THE SPECTRUM OF A FORMAL POWER SERIES RING

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**ABSTRACT.** We study the connectedness of the non-subklt locus over the spectrum of a formal power series ring. In dimension 3, we prove the existence and normality of the smallest lc centre, and apply it to the ACC for minimal log discrepancies greater than 1 on non-singular 3-folds.

## 1. INTRODUCTION

The vanishing theorem by Kodaira [17] is one of the most basic tools in algebraic geometry in characteristic zero. It is reasonable to expect a vanishing theorem on excellent schemes, but it is annoyingly unknown besides the work on surfaces by Lipman [21]. Precisely, we are interested in the relative Kodaira vanishing for a birational morphism over the spectrum of a formal power series ring  $R = K[[x_1, \dots, x_d]]$  for a field  $K$  of characteristic zero. We mean by an  $R$ -variety an integral separated scheme of finite type over  $\text{Spec } R$ .

**Conjecture 1.1.** *Let  $f: Y \rightarrow X$  be a projective birational morphism of non-singular  $R$ -varieties and  $L$  an  $f$ -ample divisor on  $Y$ . Then  $R^i f_* \mathcal{O}_Y(K_{Y/X} + L) = 0$  for  $i \geq 1$ . Here the relative canonical divisor  $K_{Y/X}$  is defined by the 0-th Fitting ideal of  $\Omega_{Y/X}$ .*

We shall not deal with this algebraic conjecture. Instead, we study the *connectedness lemma* by Shokurov [28] and Kollár [18], which is an important geometric application of the vanishing theorem in birational geometry. It claims for a proper morphism  $f: Y \rightarrow X$ , the fibrewise connectedness of the non-subklt locus of a subpair  $(Y, \Delta)$  such that  $\Delta$  is effective outside a locus in  $X$  of codimension at least 2 and such that  $-(K_Y + \Delta)$  is  $f$ -nef and  $f$ -big. We shall verify it for a germ at a non-singular point of  $X$  in the case when  $f$  is isomorphic outside the central fibre (Theorem 3.1). Investigating further in dimension 3, we obtain a desirable result on the smallest lc centre of a pair on a non-singular  $R$ -variety of dimension 3.

**Theorem 1.2.** *Let  $P \in (X, \mathfrak{a})$  be a germ of an lc but not klt pair of a non-singular  $R$ -variety  $X$  of dimension 3 and an  $\mathbb{R}$ -ideal  $\mathfrak{a}$  on  $X$ . Then the smallest lc centre of  $(X, \mathfrak{a})$  exists and it is normal.*

It is reduced to the case  $X = \text{Spec } R$  with  $K$  an algebraically closed field  $k$ . Theorem 3.1, the fibrewise connectedness, is proved by approximating the effective  $\mathbb{R}$ -divisor  $f_* \Delta$  by an  $\mathfrak{m}$ -primary  $\mathbb{R}$ -ideal  $\mathfrak{a}\langle l \rangle$ , where  $\mathfrak{m}$  is the maximal ideal sheaf, such that the non-subklt locus of the subtriplet coming from  $\mathfrak{a}\langle l \rangle$  coincides with the central fibre of the original non-subklt locus. The  $\mathfrak{a}\langle l \rangle$  is descended to  $\mathbb{A}_k^d = \text{Spec } k[x_1, \dots, x_d]$ , on which the connectedness lemma is applied. The existence of the smallest lc centre in Theorem 1.2 is a corollary to Theorem 3.1. The hardest part of Theorem 1.2 is the normality of the smallest lc centre  $C$  which is a curve. We construct an ideal sheaf  $\mathfrak{n}_C$  on the normalisation  $C_Y$  of  $C$  with  $f_C: C_Y \rightarrow C$

which satisfies  $f_{C*}\mathfrak{n}_a \subset \mathcal{O}_C$  and  $\mathcal{O}_C/f_{C*}\mathfrak{n}_a \simeq f_{C*}\mathcal{O}_{C_Y}/f_{C*}\mathfrak{n}_a$ . Then we obtain the isomorphism  $\mathcal{O}_C \simeq f_{C*}\mathcal{O}_{C_Y}$  meaning the normality of  $C$ .

Our motivation for excellent schemes stems from the notion of a generic limit of ideals due to de Fernex and Mustařă [7]. The generic limit was used to prove the ascending chain condition (ACC) for log canonical thresholds on non-singular varieties [6], the approach of which works even for the study of minimal log discrepancies [15]. We shall apply Theorem 1.2 to the ACC conjecture for minimal log discrepancies by Shokurov [27], [29] and McKernan [23] in the case of non-singular 3-folds, and settle the part of minimal log discrepancies greater than 1.

**Theorem 1.3.** *Fix subsets  $I \subset (0, \infty)$  and  $J \subset (1, 3]$  both of which satisfy the descending chain condition. Then there exist finite subsets  $I_0 \subset I$  and  $J_0 \subset J$  such that if  $P \in (X, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$  is a germ of a pair of a non-singular variety  $X$  of dimension 3 and an  $\mathbb{R}$ -ideal  $\mathfrak{a}$  on  $X$  with all  $\mathfrak{a}_j$  non-trivial at  $P$ , all  $r_j \in I$  and  $\text{mld}_P(X, \mathfrak{a}) \in J$ , then all  $r_j \in I_0$  and  $\text{mld}_P(X, \mathfrak{a}) \in J_0$ .*

The generic limit  $\mathfrak{a}$  of  $\mathbb{R}$ -ideals  $\mathfrak{a}_i$  on  $P \in X = \text{Spec } k[[x_1, \dots, x_d]]$  is an  $\mathbb{R}$ -ideal on  $P_K \in X_K = \text{Spec } K[[x_1, \dots, x_d]]$  with a field extension  $K$  of  $k$ . The ACC for minimal log discrepancies on non-singular  $d$ -folds is reduced to the stability  $\text{mld}_{P_K}(X_K, \mathfrak{a}) = \text{mld}_P(X, \mathfrak{a}_i)$  for general  $i$ . We prove it when  $(X_K, \mathfrak{a})$  is a klt pair, or even a plt pair whose lc centre has an isolated singularity, by our previous arguments [13], [14]. In dimension 3, only the case when  $(X_K, \mathfrak{a})$  has the smallest lc centre of dimension 1 remains. In this case, the estimate  $\text{mld}_{P_K}(X_K, \mathfrak{a}) \leq 1$  is derived from Theorem 1.2, which is enough to prove Theorem 1.3.

The structure of the paper is as follows. After reviewing the basics of singularities in Section 2, we study the connectedness of the non-subklt locus and establish Theorem 1.2 in Section 3. We discuss the ACC for minimal log discrepancies from the point of view of generic limits in Section 4. The stability of minimal log discrepancies in the klt and plt cases is shown in Section 5. Theorem 1.3 is completed in Section 6. The appendix exposing generic limits is attached.

Throughout this paper,  $k$  is an algebraically closed field of characteristic zero.

## 2. SINGULARITIES

We review the basics of singularities in birational geometry. A good reference is [20]. A *variety* is an integral separated scheme of finite type over  $\text{Spec } k$ . A *germ* of a scheme is considered at a closed point.

An  $\mathbb{R}$ -ideal on a noetherian scheme  $X$  is a formal product  $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$  of finitely many coherent ideal sheaves  $\mathfrak{a}_j$  on  $X$  with positive real exponents  $r_j$ . The  $\mathfrak{a}$  to the power of  $t > 0$  is  $\mathfrak{a}^t := \prod_j \mathfrak{a}_j^{tr_j}$ . The *co-support*  $\text{Cosupp } \mathfrak{a}$  of  $\mathfrak{a}$  is the union of all  $\text{Supp } \mathcal{O}_X/\mathfrak{a}_j$ . The *pull-back* of  $\mathfrak{a}$  by a morphism  $Y \rightarrow X$  is  $\mathfrak{a}\mathcal{O}_Y := \prod_j (\mathfrak{a}_j\mathcal{O}_Y)^{r_j}$ . The  $\mathbb{R}$ -ideal  $\mathfrak{a}$  is said to be *invertible* if all  $\mathfrak{a}_j$  are invertible. In this case, if in addition  $X$  is normal, then the  $\mathbb{R}$ -divisor  $A = \sum_j r_j A_j$  with  $\mathfrak{a}_j = \mathcal{O}_X(-A_j)$  is called the  $\mathbb{R}$ -divisor *defined by*  $\mathfrak{a}$ .

Let  $Z$  be an irreducible closed subset of  $X$ . We write  $\eta_Z$  for the generic point of  $Z$ . The *order* of  $\mathfrak{a}$  along  $Z$  is  $\text{ord}_Z \mathfrak{a} = \sum_j r_j \text{ord}_Z \mathfrak{a}_j$ , where  $\text{ord}_Z \mathfrak{a}_j$  is the maximal  $v \in \mathbb{N} \cup \{+\infty\}$  satisfying  $\mathfrak{a}_j\mathcal{O}_{X, \eta_Z} \subset \mathcal{I}_Z^v \mathcal{O}_{X, \eta_Z}$  for the ideal sheaf  $\mathcal{I}_Z$  of  $Z$ .

We treat a *triplet*  $(X, \Delta, \mathfrak{a})$  which consists of a normal variety  $X$ , an effective  $\mathbb{R}$ -divisor  $\Delta$  on  $X$  such that  $K_X + \Delta$  is an  $\mathbb{R}$ -Cartier  $\mathbb{R}$ -divisor, and an  $\mathbb{R}$ -ideal  $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$  on  $X$ . A prime divisor  $E$  on a normal variety  $Y$  with a birational morphism

$f: Y \rightarrow X$  is called a divisor *over*  $X$ , and the closure  $\overline{f(E)}$  of the image on  $X$  is called the *centre* of  $E$  on  $X$  and denoted by  $c_X(E)$ . We denote by  $\mathcal{D}_X$  the set of all divisors over  $X$ . The *log discrepancy* of  $E$  with respect to  $(X, \Delta, \mathfrak{a})$  is

$$a_E(X, \Delta, \mathfrak{a}) := 1 + \text{ord}_E K_{Y/(X, \Delta)} - \text{ord}_E \mathfrak{a},$$

where  $K_{Y/(X, \Delta)} := K_Y - f^*(K_X + \Delta)$  and  $\text{ord}_E \mathfrak{a} := \text{ord}_E \mathfrak{a} \mathcal{O}_Y$ . Note that  $c_X(E)$  and  $a_E(X, \Delta, \mathfrak{a})$  are determined by the valuation on the function field of  $X$  given by  $E$ .

For an irreducible closed subset  $Z$  of  $X$ , the *minimal log discrepancy* of  $(X, \Delta, \mathfrak{a})$  at  $\eta_Z$  is

$$\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) := \inf\{a_E(X, \Delta, \mathfrak{a}) \mid E \in \mathcal{D}_X, c_X(E) = Z\}.$$

It is either a non-negative real number or  $-\infty$ . We say that  $E \in \mathcal{D}_X$  *computes*  $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$  if  $c_X(E) = Z$  and  $a_E(X, \Delta, \mathfrak{a}) = \text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a})$  (or is negative when  $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) = -\infty$ ). It is often reduced to the case when  $Z$  is a closed point by the relation  $\text{mld}_{\eta_Z}(X, \Delta, \mathfrak{a}) = \text{mld}_P(X, \Delta, \mathfrak{a}) - \dim Z$  for a general closed point  $P \in Z$  (cf. [3, Proposition 2.1]).

The triplet  $(X, \Delta, \mathfrak{a})$  is said to be *log canonical (lc)* (resp. *Kawamata log terminal (klt)*) if  $a_E(X, \Delta, \mathfrak{a}) \geq 0$  (resp.  $> 0$ ) for all  $E \in \mathcal{D}_X$ . It is said to be *purely log terminal (plt)* (resp. *canonical, terminal*) if  $a_E(X, \Delta, \mathfrak{a}) > 0$  (resp.  $\geq 1, > 1$ ) for all exceptional  $E \in \mathcal{D}_X$ . The log canonicity of  $(X, \Delta, \mathfrak{a})$  about  $P \in X$  is equivalent to  $\text{mld}_P(X, \Delta, \mathfrak{a}) \geq 0$ . Let  $Y$  be a normal variety with a birational morphism to  $X$ . A centre  $c_Y(E)$  with  $a_E(X, \Delta, \mathfrak{a}) \leq 0$  is called a *non-klt centre* on  $Y$  of  $(X, \Delta, \mathfrak{a})$ . The union of all non-klt centres on  $Y$  is called the *non-klt locus* on  $Y$  and denoted by  $\text{Nklt}_Y(X, \Delta, \mathfrak{a})$ . When we say just a non-klt centre or the non-klt locus, we mean that it is on  $X$ .

A *log resolution* of  $(X, \Delta, \mathfrak{a})$  is a projective morphism  $f: Y \rightarrow X$  from a non-singular variety  $Y$  such that (i)  $\text{Exc } f$  is a divisor, (ii)  $\mathfrak{a} \mathcal{O}_Y$  is invertible, (iii)  $\text{Exc } f \cup \text{Supp } \Delta_Y \cup \text{Cosupp } \mathfrak{a} \mathcal{O}_Y$  is a simple normal crossing (snc) divisor, where  $\Delta_Y$  is the strict transform of  $\Delta$ , and (iv)  $f$  is isomorphic on the locus  $U$  in  $X$  with  $U$  non-singular,  $\mathfrak{a}|_U$  invertible and  $\text{Supp } \Delta|_U \cup \text{Cosupp } \mathfrak{a}|_U$  snc. A *stratum* (resp. an *open stratum*) of an snc divisor  $\sum_{i \in I} E_i$  is an irreducible component of  $\bigcap_{i \in J} E_i$  (resp.  $\bigcap_{i \in J} E_i \setminus \bigcup_{i \notin J} E_i$ ) for a subset  $J$  of  $I$ .

By allowing a not necessarily effective  $\mathbb{R}$ -divisor  $\Delta$ , one can consider a *sub-triplet*  $(X, \Delta, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$ . The notions of lc (resp. klt) singularities are extended for subtriplets, in which we say *sublc* (resp. *subklt*) singularities. Let  $f: Y \rightarrow X$  be a birational morphism from a non-singular variety  $Y$  such that  $\text{Exc } f$  is a divisor  $\sum_i E_i$ . The *weak transform* on  $Y$  of  $\mathfrak{a}$  is the  $\mathbb{R}$ -ideal  $\mathfrak{a}_Y = \prod_j \mathfrak{a}_{jY}^{r_j}$  with  $\mathfrak{a}_{jY} = \mathfrak{a}_j \mathcal{O}_Y(\sum_i (\text{ord}_{E_i} \mathfrak{a}_j) E_i)$ .

**Definition 2.1.** Notation as above. The *pull-back* of  $(X, \Delta, \mathfrak{a})$  by  $f$  is the subtriplet  $(Y, \Delta_Y, \mathfrak{a}_Y)$  where  $\Delta_Y = -K_{Y/(X, \Delta)} + \sum_{ij} (r_j \text{ord}_{E_i} \mathfrak{a}_j) E_i$ .

The  $(X, \Delta, \mathfrak{a})$  is sublc (resp. subklt) if and only if so is  $(Y, \Delta_Y, \mathfrak{a}_Y)$ . We use the notation  $\text{Nklt}_Y(X, \Delta, \mathfrak{a})$  also for the non-subklt locus on  $Y$  of a subtriplet  $(X, \Delta, \mathfrak{a})$ .

These definitions are extended on schemes over a field  $K$  of characteristic zero and even over a formal power series ring  $R = K[[x_1, \dots, x_d]]$  by the existence of log resolutions due to Hironaka [11] and Temkin [31], [32]. This extension is studied by de Fernex, Ein and Mustařă [6], [7]. We mean by an *R-variety* an integral separated scheme of finite type over  $\text{Spec } R$ .

The canonical divisor  $K_X$  on a normal  $R$ -variety  $X$  is defined by the isomorphism  $\mathcal{O}_X(K_X)|_U \simeq \wedge^r \Omega'_{X/K}|_U$  on the non-singular locus  $U$  of  $X$ , where  $\Omega'_{X/K}$  is the *sheaf of special differentials* in [6] and  $r$  is its rank. The relative canonical divisor is well understood for a birational morphism of non-singular  $R$ -varieties.

**Lemma 2.2** ([6, Remark A.12]). *Let  $Y \rightarrow X$  be a proper birational morphism of non-singular  $R$ -varieties. Then  $K_{Y/X}$  is the effective divisor defined by the 0-th Fitting ideal of  $\Omega_{Y/X}$ . In particular,  $K_{Y/X}$  is independent of the structure of  $X$  as an  $R$ -variety.*

The log discrepancies are preserved by field extensions and completions.

**Corollary 2.3.** *Let  $Y \rightarrow X$  be as in Lemma 2.2. Take an  $R'$ -variety  $X'$  as in (i), (ii) or (iii) below and set a morphism  $Y' = Y \times_X X' \rightarrow X'$  of  $R'$ -varieties.*

- (i)  $X'$  is a component of  $X \times_{\text{Spec } R} \text{Spec } R'$  with  $R' = \widehat{R \otimes_K K'}$  for a field extension  $K'$  of  $K$ .
- (ii)  $X' = \widehat{\text{Spec } \mathcal{O}_{X,P}}$  for a germ  $P \in X$ , which admits the structure of an  $R'$ -variety for a suitable  $R' = K'[[x_1, \dots, x_d]]$  by Cohen's structure theorem [4].
- (iii)  $X' = X$  with another structure morphism  $X \rightarrow \text{Spec } R'$ .

Then  $K_{Y'/X'}$  is the pull-back of  $K_{Y/X}$ . In particular, for an  $\mathbb{R}$ -ideal  $\mathfrak{a}$  on  $X$ , a divisor  $E$  over  $X$  and a germ  $P \in X$ , one has  $a_{E'}(X', \mathfrak{a}_{X'}) = a_E(X, \mathfrak{a})$  for a component  $E'$  of  $E \times_X X'$  and  $\text{mld}_{P'}(X', \mathfrak{a}_{X'}) = \text{mld}_P(X, \mathfrak{a})$  for a point  $P'$  of  $P \times_X X'$ .

This is by the regularity of the morphism  $X' \rightarrow X$ . The cases (i) and (ii) for  $R = K$  are stated in [6, Lemma 2.14, Propositions 2.11, A.14] even for a normal ( $\mathbb{Q}$ -Gorenstein)  $K$ -variety  $X$ .

Suppose that  $(X, \Delta, \mathfrak{a})$  is lc. Then a non-klt centre (on  $X$ ) of  $(X, \Delta, \mathfrak{a})$  is often called an *lc centre*. An lc centre which is minimal with respect to inclusions is called a *minimal lc centre*. When we work over a germ  $P \in X$ , the following definition makes sense.

**Definition 2.4.** Let  $P \in (X, \Delta, \mathfrak{a})$  be a germ of an lc triplet. The *smallest lc centre* is an lc centre of  $(X, \Delta, \mathfrak{a})$  passing through  $P$  contained in every lc centre passing through  $P$ .

If  $X$  is a variety, then the smallest lc centre exists and it is normal [8, Theorem 9.1]. It is, however, unknown for  $R$ -varieties. Theorem 1.2 states that this is the case when  $X$  is a non-singular  $R$ -variety of dimension 3.

### 3. THE SMALLEST LC CENTRE ON A THREEFOLD

This section is devoted to the proof of Theorem 1.2. We work over a germ  $P \in X$  of an  $R$ -variety with  $R = K[[x_1, \dots, x_d]]$ . The maximal ideal sheaf of  $P \in X$  is denoted by  $\mathfrak{m}$ . When we discuss on the spectrum of a noetherian ring, we identify an ideal in the ring with its coherent ideal sheaf.

**3.A. A connectedness theorem.** We prove a connectedness theorem over  $X$ .

**Theorem 3.1.** *Let  $P \in (X, \mathfrak{a})$  be a germ of a pair on a non-singular  $R$ -variety  $X$  and  $f: Y \rightarrow X$  a proper birational morphism of non-singular  $R$ -varieties which is isomorphic outside  $P$ . Let  $\Delta$  be an  $\mathbb{R}$ -divisor on  $Y$  with  $f_*\Delta \geq 0$  such that  $-(K_Y + \Delta)$  is  $f$ -nef. Then  $\text{Nklt}_Y(Y, \Delta, \mathfrak{a}_{\mathcal{O}_Y}) \cap f^{-1}(P)$  is connected.*

We extract the case  $\Delta = -K_{Y/X}$ .

**Corollary 3.2.** *Let  $P \in (X, \mathfrak{a})$  be a germ of a pair on a non-singular  $R$ -variety  $X$  and  $f: Y \rightarrow X$  a proper birational morphism of non-singular  $R$ -varieties which is isomorphic outside  $P$ . Then  $\text{Nklt}_Y(X, \mathfrak{a}) \cap f^{-1}(P)$  is connected.*

The statement for  $R = k$  is a special case of the connectedness lemma by Shokurov and Kollár [18, Theorem 17.4]. It settles the case when  $\mathfrak{a}$  is  $\mathfrak{m}$ -primary and  $\Delta$  is  $f$ -exceptional. Write  $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ .

**Lemma 3.3.** (i) *In order to prove Theorem 3.1, one may assume that  $X = \text{Spec } R$  with  $K = k$ ,  $f$  is projective and  $\Delta$  is  $f$ -exceptional.*  
(ii) *Theorem 3.1 holds in the case when  $X = \text{Spec } R$  with  $K = k$ ,  $f$  is projective,  $\Delta$  is  $f$ -exceptional and all  $\mathfrak{a}_j$  are  $\mathfrak{m}$ -primary ideals.*

*Proof.* (i) Take an isomorphism  $\widehat{\mathcal{O}_{X,P}} \simeq K'[[x_1, \dots, x_d]]$  with  $K' = \mathcal{O}_{X,P}/\mathfrak{m}$  by Cohen's structure theorem and set  $R' = k[[x_1, \dots, x_d]]$  for the algebraic closure  $k$  of  $K'$ . Because the base change  $\text{Spec } R' \rightarrow X$  commutes with taking the non-subklt locus by Corollary 2.3, we may assume  $X = \text{Spec } R$  with  $K = k$  (the  $d$  may be changed). By the flattening theorem of Raynaud and Gruson [25, Théorème 1<sup>re</sup> 5.2.2], there exists a projective morphism  $f': Y' \rightarrow X$  from a non-singular  $R$ -variety  $Y'$  which is isomorphic outside  $P$  and factors through  $f$ . Replacing  $(Y, \Delta)$  with its pull-back on  $Y'$ , we may assume that  $f$  is projective. The  $\Delta' := \Delta - f^* f_* \Delta$  is  $f$ -exceptional. Take an invertible  $\mathbb{R}$ -ideal  $\mathfrak{d}$  on  $X$  which defines the  $\mathbb{R}$ -divisor  $f_* \Delta \geq 0$ . Then  $\text{Nklt}_Y(Y, \Delta, \mathfrak{a} \mathcal{O}_Y) = \text{Nklt}_Y(Y, \Delta', \mathfrak{a} \mathfrak{d} \mathcal{O}_Y)$ . Replacing  $\Delta$  with  $\Delta'$  and  $\mathfrak{a}$  with  $\mathfrak{a} \mathfrak{d}$ , we may assume that  $\Delta$  is  $f$ -exceptional.

(ii) We use the notation  $\bar{R} = k[x_1, \dots, x_d]$  and  $\mathbb{A}_k^d = \text{Spec } \bar{R}$  with origin  $\bar{P}$ . By Proposition A.7,  $f$  is the base change of a projective morphism  $\bar{f}: \bar{Y} \rightarrow \mathbb{A}_k^d$  and  $\mathfrak{a}$  is the pull-back of the  $\mathbb{R}$ -ideal  $\bar{\mathfrak{a}} = \prod_j (\mathfrak{a}_j \cap \bar{R})^{r_j}$ . Then  $f^{-1}(P) \simeq \bar{f}^{-1}(\bar{P})$  and  $\Delta$  is the base change of an  $\bar{f}$ -exceptional  $\mathbb{R}$ -divisor  $\bar{\Delta}$  such that  $-(K_{\bar{Y}} + \bar{\Delta})$  is  $\bar{f}$ -nef. Thus  $f^{-1}(P) \supset \text{Nklt}_Y(Y, \Delta, \mathfrak{a} \mathcal{O}_Y) \simeq \text{Nklt}_{\bar{Y}}(\bar{Y}, \bar{\Delta}, \bar{\mathfrak{a}} \mathcal{O}_{\bar{Y}})$ , which is connected by [18, Theorem 17.4]. q.e.d.

We take a log resolution  $q: W \rightarrow Y$  of  $(Y, \Delta, \mathfrak{a} \mathcal{O}_Y)$  and set the composition  $g = f \circ q: W \rightarrow X$ . We fix  $\varepsilon > 0$  such that

$$(1) \quad F := \text{Nklt}_W(Y, \Delta, \mathfrak{a} \mathcal{O}_Y) = \text{Nklt}_W(Y, \Delta, \mathfrak{a}^{1+\varepsilon} \mathcal{O}_Y).$$

We approximate  $\mathfrak{a}$  by an  $\mathfrak{m}$ -primary  $\mathbb{R}$ -ideal

$$(2) \quad \mathfrak{a}\langle l \rangle := \prod_j (\mathfrak{a}_j + \mathfrak{m}^l)^{r_j(1+\varepsilon)}$$

with  $l \in \mathbb{N}$ .

Consider an irreducible component  $D$  of  $F \cap g^{-1}(P)$  with  $\text{codim}_W D = 2$ , and let  $E_D \subset g^{-1}(P)$  and  $F_D \subset F$  be the prime divisors such that  $D \subset E_D \cap F_D$ . We build a tower of blow-ups

$$(3) \quad \cdots \rightarrow W_i \xrightarrow{g_i} W_{i-1} \rightarrow \cdots \rightarrow W_0 = W$$

as follows. Set  $W_0 := W$ ,  $E_0 := E_D$  and  $F_0 := F_D$ . We construct inductively the blow-up  $g_i: W_i \rightarrow W_{i-1}$  along  $D$  for  $i = 1$  (resp. along  $E_{i-1} \cap F_{i-1}$  for  $i \geq 2$ ), and set  $E_i$  as the exceptional divisor of  $g_i$ , and  $F_i$  as the strict transform on  $W_i$  of  $F_D$ . The composition  $g_1 \circ \cdots \circ g_i$  is denoted by  $h_i: W_i \rightarrow W$ .

**Lemma 3.4.** (i)  $a_{E_i}(Y, \Delta, \mathfrak{a}^{1+\varepsilon} \mathcal{O}_Y) \leq a_{E_D}(Y, \Delta, \mathfrak{a}^{1+\varepsilon} \mathcal{O}_Y) - i\varepsilon \operatorname{ord}_{F_D} \mathfrak{a}$ .  
(ii)  $h_{i*} \mathcal{O}_{W_i}(-aE_i) \subset \mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D)$  for any  $a \in \mathbb{N}$ .

*Proof.* The (i) is just a computation using  $a_{F_D}(Y, \Delta, \mathfrak{a} \mathcal{O}_Y) \leq 0$ . The (ii) is from  $h_{i*} \mathcal{O}_{W_i}(-aE_i) \cdot \mathcal{O}_{F_D} \subset h_{i*} \mathcal{O}_{F_i}(-aE_i|_{F_i}) = \mathcal{O}_{F_D}(-aE_D|_{F_D})$  via  $F_i \simeq F_D$ . q.e.d.

**Lemma 3.5.** Suppose that  $(Y, \Delta)$  is klt outside  $f^{-1}(P)$ . Then there exists  $l$  such that  $\operatorname{Nklt}_W(Y, \Delta, \mathfrak{a}\langle l \rangle \mathcal{O}_Y) = F \cap g^{-1}(P)$ .

*Proof.* By (1), (2) and the assumption,  $\operatorname{Nklt}_W(Y, \Delta, \mathfrak{a}\langle l \rangle \mathcal{O}_Y) \subset F \cap g^{-1}(P)$  for any  $l$ . Thus it suffices to prove that for every irreducible component  $D$  of  $F \cap g^{-1}(P)$ , there exists  $l_D$  such that  $D$  is a non-subklt centre on  $W$  of  $(Y, \Delta, \mathfrak{a}\langle l \rangle \mathcal{O}_Y)$  for any  $l \geq l_D$ . If  $\operatorname{codim}_W D = 1$ , then we may take any  $l_D$  such that  $l_D \operatorname{ord}_D \mathfrak{m} \geq \operatorname{ord}_D \mathfrak{a}_j$  for all  $j$ . If  $\operatorname{codim}_W D = 2$ , then  $\operatorname{ord}_{F_D} \mathfrak{a} > 0$  and we take the tower of blow-ups in (3). By Lemma 3.4(i), we have  $a_{E_i}(Y, \Delta, \mathfrak{a}^{1+\varepsilon} \mathcal{O}_Y) \leq 0$  whenever  $a_{E_D}(Y, \Delta, \mathfrak{a}^{1+\varepsilon} \mathcal{O}_Y) \leq i\varepsilon \operatorname{ord}_{F_D} \mathfrak{a}$ . Fix such  $i$  and take  $l_D$  such that  $l_D \operatorname{ord}_{E_i} \mathfrak{m} \geq \operatorname{ord}_{E_i} \mathfrak{a}_j$  for all  $j$ . Then for  $l \geq l_D$ ,  $a_{E_i}(Y, \Delta, \mathfrak{a}\langle l \rangle \mathcal{O}_Y) = a_{E_i}(Y, \Delta, \mathfrak{a}^{1+\varepsilon} \mathcal{O}_Y) \leq 0$ , so  $D = c_W(E_i)$  is a non-subklt centre on  $W$  of  $(Y, \Delta, \mathfrak{a}\langle l \rangle \mathcal{O}_Y)$ . q.e.d.

*Proof of Theorem 3.1.* After the reduction in Lemma 3.3(i), we take  $l$  in Lemma 3.5. Then  $\operatorname{Nklt}_Y(Y, \Delta, \mathfrak{a} \mathcal{O}_Y) \cap f^{-1}(P) = q(F \cap g^{-1}(P)) = q(\operatorname{Nklt}_W(Y, \Delta, \mathfrak{a}\langle l \rangle \mathcal{O}_Y)) = \operatorname{Nklt}_Y(Y, \Delta, \mathfrak{a}\langle l \rangle \mathcal{O}_Y)$ . Apply Lemma 3.3(ii) to  $(Y, \Delta, \mathfrak{a}\langle l \rangle \mathcal{O}_Y)$ . q.e.d.

In our proof of Theorem 3.1, we do not know a relative vanishing for  $g: W \rightarrow X$ . Instead, we consider a log resolution  $f_l: Y_l \rightarrow X$  of  $(X, \mathfrak{a}\langle l \rangle \mathfrak{m})$  which factors through  $f$ , and let  $p_l: Y_l \rightarrow Y$  be the induced morphism. The  $l$  is not fixed here. The  $f_l$  is isomorphic outside  $P$ . Let  $(Y_l, \Delta_l, \mathcal{O}_{Y_l})$  be the pull-back of  $(X, 0, \mathfrak{a}\langle l \rangle)$ . Then we have a vanishing involving  $\Delta_l$ .

**Lemma 3.6.** Let  $f_l = f \circ p_l: Y_l \rightarrow Y \rightarrow X$  be as above. Write  $[-\Delta_l] = P_l - N_l$  by effective divisors  $P_l$  and  $N_l$  with no common divisors. Then

$$R^1 f_{l*} (p_{l*} \mathcal{O}_{Y_l}(-N_l)) = 0.$$

*Proof.* The sheaf  $R^1 f_{l*} (p_{l*} \mathcal{O}_{Y_l}(-N_l))$  is supported in  $P$ . Set  $\widehat{\mathcal{O}_{X,P}} \simeq K'[[x_1, \dots, x_{d'}]]$  and  $R' = k[[x_1, \dots, x_{d'}]]$  for the algebraic closure  $k$  of  $K'$ , then  $R'$  is faithfully flat over  $\mathcal{O}_{X,P}$ . Hence taking the base change to  $\operatorname{Spec} R'$ , one can reduce to the case  $X = \operatorname{Spec} R$  with  $K = k$  by [9, Proposition III.1.4.15] and Corollary 2.3. By Proposition A.7,  $f_l$  is the base change of a projective morphism  $\tilde{f}_l: \tilde{Y}_l \rightarrow \mathbb{A}_k^d$ . The  $\mathfrak{a}\langle l \rangle$  is the pull-back of an  $\mathbb{R}$ -ideal  $\tilde{\mathfrak{a}}\langle l \rangle$  on  $\mathbb{A}_k^d$ , and  $\Delta_l$  is the base change of the  $\mathbb{R}$ -divisor  $\tilde{\Delta}_l$  on  $\tilde{Y}_l$  such that  $(\tilde{Y}_l, \tilde{\Delta}_l, \mathcal{O}_{\tilde{Y}_l})$  is the pull-back of  $(\mathbb{A}_k^d, 0, \tilde{\mathfrak{a}}\langle l \rangle)$ .

Kawamata–Viehweg vanishing theorem [16], [34] implies  $R^1 \tilde{f}_{l*} \mathcal{O}_{\tilde{Y}_l}(\lceil -\tilde{\Delta}_l \rceil) = 0$ . Since  $X \rightarrow \mathbb{A}_k^d$  is flat, this is base-changed to  $R^1 f_{l*} \mathcal{O}_{Y_l}(\lceil -\Delta_l \rceil) = 0$  by [9, Proposition III.1.4.15]. Thus, applying  $f_{l*}$  to the exact sequence

$$0 \rightarrow \mathcal{O}_{Y_l}(P_l - N_l) \rightarrow \mathcal{O}_{Y_l}(P_l) \rightarrow \mathcal{O}_{N_l}(P_l|_{N_l}) \rightarrow 0,$$

we obtain the surjection  $\mathcal{O}_X = f_{l*} \mathcal{O}_{Y_l}(P_l) \twoheadrightarrow f_{l*} \mathcal{O}_{N_l}(P_l|_{N_l})$ . This homomorphism is factored as  $\mathcal{O}_X \rightarrow f_{l*} \mathcal{O}_{N_l} \hookrightarrow f_{l*} \mathcal{O}_{N_l}(P_l|_{N_l})$ , so we have the surjection  $\mathcal{O}_X \twoheadrightarrow f_{l*} \mathcal{O}_{N_l}$ . Moreover, we have the base change  $R^1 f_{l*} \mathcal{O}_{Y_l} = 0$  of the vanishing  $R^1 \tilde{f}_{l*} \mathcal{O}_{\tilde{Y}_l} = 0$ . Hence applying  $f_{l*}$  to the exact sequence

$$0 \rightarrow \mathcal{O}_{Y_l}(-N_l) \rightarrow \mathcal{O}_{Y_l} \rightarrow \mathcal{O}_{N_l} \rightarrow 0,$$

we obtain  $R^1 f_{l*} \mathcal{O}_{Y_l}(-N_l) = 0$ .

Leray spectral sequence  $R^p f_* (R^q p_{l*} \mathcal{O}_{Y_l}(-N_l)) \Rightarrow R^{p+q} f_{l*} \mathcal{O}_{Y_l}(-N_l)$  gives an injection  $R^1 f_* (p_{l*} \mathcal{O}_{Y_l}(-N_l)) \hookrightarrow R^1 f_{l*} \mathcal{O}_{Y_l}(-N_l)$ , so  $R^1 f_* (p_{l*} \mathcal{O}_{Y_l}(-N_l)) = 0$ . q.e.d.

**3.B. Propositions in an arbitrary dimension.** We prepare two auxiliary propositions.

It is easy to see that a minimal lc centre of codimension 1 is normal.

**Proposition 3.7.** *Let  $(X, \mathfrak{a})$  be a pair on a non-singular  $R$ -variety  $X$ , and  $S$  the union of all non-klt centres of codimension 1 of  $(X, \mathfrak{a})$ . Then every irreducible component of the non-normal locus of  $S$  is a non-klt centre of  $(X, \mathfrak{a})$ .*

*Proof.* Since  $S$  is Cohen–Macaulay, an irreducible component  $C$  of the non-normal locus of  $S$  has  $\text{codim}_X C = 2$  and  $\text{mult}_{\eta_C} S \geq 2$ . Let  $E$  be the divisor over  $X$  obtained at  $\eta_C$  by the blow-up of  $X$  along  $C$ . Then  $a_E(X, \mathfrak{a}) = 2 - \text{ord}_E \mathfrak{a} \leq 2 - \text{mult}_{\eta_C} S \leq 0$ , so  $C = c_X(E)$  is a non-klt centre of  $(X, \mathfrak{a})$ . q.e.d.

We can perturb  $\mathfrak{a}$  to reduce to the case when every lc centre is minimal.

**Proposition 3.8.** *Let  $(X, \mathfrak{a})$  be an lc pair on a klt  $R$ -variety  $X$ . Then there exists an  $\mathbb{R}$ -ideal  $\mathfrak{a}'$  forming an lc pair  $(X, \mathfrak{a}')$  such that a minimal lc centre of  $(X, \mathfrak{a})$  is an lc centre of  $(X, \mathfrak{a}')$  and vice versa.*

*Proof.* Let  $\{Z_i\}_i$  be the set of all minimal lc centres of  $(X, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$ . For each  $Z_i$ , fix  $E_i \in \mathcal{D}_X$  computing  $\text{mld}_{\eta_{Z_i}}(X, \mathfrak{a}) = 0$ . Let  $\mathcal{J}_Z$  be the ideal sheaf of  $Z = \bigcup_i Z_i$ , and take an integer  $l$  such that  $l \text{ord}_{E_i} \mathcal{J}_Z \geq \text{ord}_{E_i} \mathfrak{a}_j$  for all  $i, j$ . Then  $(X, \mathfrak{a}' := \prod_j (\mathfrak{a}_j + \mathcal{J}_Z^l)^{r_j})$  is lc, and  $Z_i$  is an lc centre of  $(X, \mathfrak{a}')$  by  $\text{ord}_{E_i} \mathfrak{a}' = \text{ord}_{E_i} \mathfrak{a}$ . On the other hand, every lc centre of  $(X, \mathfrak{a}')$  is an lc centre of  $(X, \mathfrak{a})$  contained in  $\text{Cosupp } \mathfrak{a}' = Z$ , so it equals some  $Z_i$ . q.e.d.

**3.C. The smallest lc centre on a threefold.** We proceed to the proof of Theorem 1.2. We may assume that  $P$  is not an lc centre of  $(X, \mathfrak{a})$ . By Proposition 3.8, we may assume that every lc centre of  $(X, \mathfrak{a})$  is minimal.

The existence of the smallest lc centre is a consequence of Corollary 3.2.

*Proof of the existence of the smallest lc centre.* Let  $\{Z_i\}_i$  be the set of all lc centres of  $(X, \mathfrak{a})$ , which are assumed to be minimal. Proposition 3.7 implies that  $Z = \bigcup_i Z_i$  is non-singular outside  $P$ . Thus we have an embedded resolution  $f: Y \rightarrow X$  of singularities of  $Z$ , in which  $f$  is isomorphic outside  $P$  and induces  $f_Z: \bigsqcup_i Z_{iY} \rightarrow Z$  for the strict transform  $Z_{iY}$  of  $Z_i$ . By Corollary 3.2,  $f_Z^{-1}(P) = \text{Nklt}_Y(X, \mathfrak{a}) \cap f^{-1}(P)$  is connected, that is, there exists only one lc centre of  $(X, \mathfrak{a})$ . q.e.d.

*Remark 3.9.* The above proof shows that if  $Z$  is the smallest lc centre of  $(X, \mathfrak{a})$ , then its normalisation  $Z^\vee \rightarrow Z$  is a homeomorphism.

To complete Theorem 1.2, we must prove that the unique lc centre of  $(X, \mathfrak{a})$  is normal. If it is a surface, then it is normal by Proposition 3.7. Thus, we may assume that  $(X, \mathfrak{a})$  has the unique lc centre  $C$  which is a curve. We have an embedded resolution  $f: Y \rightarrow X$  of singularities of  $C$ , in which  $f$  is isomorphic outside  $P$  and induces the normalisation  $f_C: C_Y \rightarrow C$  for the strict transform  $C_Y$  of  $C$ . Note that  $f_C^{-1}(P)$  consists of one point, say  $P_Y$ , by Remark 3.9. We let  $\mathfrak{n}$  denote the maximal ideal sheaf of  $P_Y \in Y$ . Then we take a log resolution  $q: W \rightarrow Y$  of  $(Y, \text{am } \mathcal{O}_Y \cdot \mathfrak{n})$  and set the composition  $g = f \circ q: W \rightarrow X$ .

We fix  $\varepsilon$  in (1) for  $\Delta = -K_{Y/X}$ , that is,  $F = \text{Nklt}_W(X, \mathfrak{a}) = \text{Nklt}_W(X, \mathfrak{a}^{1+\varepsilon})$ . For the  $\mathfrak{a}\langle l \rangle$  in (2), we consider a log resolution  $f_l: Y_l \rightarrow X$  of  $(X, \mathfrak{a}\langle l \rangle \mathfrak{m})$  which factors through  $f$  as  $f_l = f \circ p_l$ . We extend Lemma 3.6.

**Lemma 3.10.** *Let  $f$  and  $f_l = f \circ p_l$  be as above. Then for an arbitrary ideal sheaf  $\mathcal{I}$  on  $Y$  containing  $p_{l*}\mathcal{O}_{Y_l}(-N_l)$ , with  $N_l$  in Lemma 3.6, one has  $R^1 f_* \mathcal{I} = 0$ .*

*Proof.* By (1) for  $\Delta = -K_{Y/X}$  and (2), we see  $p_l(\text{Supp } N_l) = \text{Nklt}_Y(X, \mathfrak{a}\langle l \rangle) \subset q(F \cap g^{-1}(P)) = C_Y \cap f^{-1}(P) = P_Y$ , whence the cokernel  $\mathcal{Q}$  of the natural injection  $p_{l*}\mathcal{O}_{Y_l}(-N_l) \hookrightarrow \mathcal{I}$  is a skyscraper sheaf. In particular,  $R^1 f_* \mathcal{Q} = 0$ . Apply  $f_*$  to the exact sequence

$$0 \rightarrow p_{l*}\mathcal{O}_{Y_l}(-N_l) \rightarrow \mathcal{I} \rightarrow \mathcal{Q} \rightarrow 0.$$

By Lemma 3.6 and  $R^1 f_* \mathcal{Q} = 0$ , we obtain  $R^1 f_* \mathcal{I} = 0$ . q.e.d.

We fix an irreducible component  $D$  of  $F \cap q^{-1}(P_Y)$ , which is a curve, and let  $E_D \subset q^{-1}(P_Y)$  and  $F_D \subset F$  be the prime divisors such that  $D \subset E_D \cap F_D$ . We derive a vanishing for ideal sheaves on  $Y$  close to that of  $C_Y$ .

**Lemma 3.11.**  *$R^1 f_*(q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D))) = 0$  for any  $a \in \mathbb{N}$ .*

*Proof.* Take the tower of blow-ups in (3). For fixed  $a$ , choose  $i \in \mathbb{N}$  such that  $a_{E_D}(X, \mathfrak{a}^{1+\varepsilon}) - i\varepsilon \text{ord}_{F_D} \mathfrak{a} \leq -a$ . Then Lemma 3.4 for  $\Delta = -K_{Y/X}$  shows

$$(4) \quad h_{i*}\mathcal{O}_{W_i}([a_{E_i}(X, \mathfrak{a}^{1+\varepsilon})]E_i) \subset h_{i*}\mathcal{O}_{W_i}(-aE_i) \subset \mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D).$$

Take  $l$  such that  $l \text{ord}_{E_i} \mathfrak{m} \geq \text{ord}_{E_i} \mathfrak{a}_j$  for all  $j$ . Then,

$$(5) \quad a_{E_i}(X, \mathfrak{a}^{1+\varepsilon}) = a_{E_i}(X, \mathfrak{a}\langle l \rangle).$$

For this  $l$ , we take a log resolution  $f_l: Y_l \rightarrow X$  of  $(X, \mathfrak{a}\langle l \rangle \mathfrak{m})$  which factors through  $f$ , such that  $c_{Y_l}(E_i)$  is a divisor. Then by (4) and (5), one can apply Lemma 3.10 to  $\mathcal{I} = q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D))$ . q.e.d.

We set the ideal sheaf  $\mathfrak{n}_a$  on  $C_Y$  as

$$\mathfrak{n}_a := q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D)) \cdot \mathcal{O}_{C_Y}.$$

**Lemma 3.12.** *There exists  $a$  such that  $f_{C*}\mathfrak{n}_a \subset \mathcal{O}_C$ .*

*Proof.* Note that  $\mathfrak{n}_{C_Y}$  is an invertible ideal sheaf on  $C_Y$ . Set  $n = \text{ord}_{E_D} \mathfrak{n}$ , then

$$(6) \quad \mathfrak{n}_{nl} \subset q_*\mathcal{O}_{F_D}(-nlE_D|_{F_D}) = \mathfrak{n}^l \mathcal{O}_{C_Y}$$

for any  $l$ . Take an  $f$ -exceptional divisor  $A \geq 0$  on  $Y$  such that  $-A$  is  $f$ -ample and set  $\mathcal{O}_{C_Y}(-A|_{C_Y}) = \mathfrak{n}^t \mathcal{O}_{C_Y}$ . By Serre vanishing theorem [9, Théorème III.2.2.1], there exists  $m_0$  such that  $R^1 f_* \mathcal{I}_{C_Y}(-mA) = 0$  for any  $m \geq m_0$ , where  $\mathcal{I}_{C_Y}$  is the ideal sheaf of  $C_Y$  on  $Y$ . Then we have the surjection  $f_* \mathcal{O}_Y(-mA) \twoheadrightarrow f_{C*} \mathcal{O}_{C_Y}(-mA|_{C_Y}) = f_{C*} \mathfrak{n}^{tm} \mathcal{O}_{C_Y}$ , which provides

$$(7) \quad f_{C*} \mathfrak{n}^{tm} \mathcal{O}_{C_Y} = f_* \mathcal{O}_Y(-mA) \cdot \mathcal{O}_C \subset \mathcal{O}_C.$$

Combining (6) and (7), we obtain  $f_{C*} \mathfrak{n}_{ntm} \subset f_{C*} \mathfrak{n}^{tm} \mathcal{O}_{C_Y} \subset \mathcal{O}_C$  for  $m \geq m_0$ . q.e.d.

*Proof of the normality of  $C$ .* Applying  $f_*$  to the exact sequence

$$0 \rightarrow q_*(\mathcal{O}_W(-aE_D) + \mathcal{O}_W(-F_D)) \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_{C_Y}/\mathfrak{n}_a \rightarrow 0$$

and using Lemma 3.11, we obtain the surjection  $\mathcal{O}_X \twoheadrightarrow f_{C*}(\mathcal{O}_{C_Y}/\mathfrak{n}_a)$ . This homomorphism is factored as  $\mathcal{O}_X \twoheadrightarrow \mathcal{O}_C/f_{C*}\mathfrak{n}_a \cap \mathcal{O}_C \hookrightarrow f_{C*}\mathcal{O}_{C_Y}/f_{C*}\mathfrak{n}_a \hookrightarrow f_{C*}(\mathcal{O}_{C_Y}/\mathfrak{n}_a)$ ,



so we have an isomorphism  $\mathcal{O}_C/f_{C*}\mathfrak{n}_a \cap \mathcal{O}_C \simeq f_{C*}\mathcal{O}_{C_Y}/f_{C*}\mathfrak{n}_a$ . For  $a$  in Lemma 3.12, it is  $\mathcal{O}_C/f_{C*}\mathfrak{n}_a \simeq f_{C*}\mathcal{O}_{C_Y}/f_{C*}\mathfrak{n}_a$ . Therefore  $\mathcal{O}_C \simeq f_{C*}\mathcal{O}_{C_Y}$ , meaning the normality of  $C$ . q.e.d.

Theorem 1.2 is established.

*Remark 3.13.* (i) One may prove the normality of  $C$  by using Zariski's subspace theorem [1, (10.6)]. One has an isomorphism  $\mathcal{O}_C/f_{C*}\mathfrak{n}_a \cap \mathcal{O}_C \simeq f_{C*}(\mathcal{O}_{C_Y}/\mathfrak{n}_a)$  for any  $a$ . By (6), the family  $\{\mathfrak{n}_a\}_a$  gives the  $\mathfrak{n}\mathcal{O}_{C_Y}$ -adic topology. Since the family  $\{f_*\mathcal{O}_Y(-mA)\}_m$  in the proof of Lemma 3.12 gives the  $\mathfrak{m}$ -adic topology by Zariski's subspace theorem (cf. [12, Lemma 3]), we see from (7) that the family  $\{f_{C*}\mathfrak{n}_a \cap \mathcal{O}_C\}_a$  as well as  $\{f_{C*}\mathfrak{n}^a\mathcal{O}_{C_Y} \cap \mathcal{O}_C\}_a$  gives the  $\mathfrak{m}\mathcal{O}_C$ -adic topology. Hence  $\widehat{\mathcal{O}_{C,P}} \simeq \varprojlim_a \mathcal{O}_C/f_{C*}\mathfrak{n}_a \cap \mathcal{O}_C \simeq \varprojlim_a f_{C*}(\mathcal{O}_{C_Y}/\mathfrak{n}_a) \simeq \widehat{\mathcal{O}_{C_Y,P_Y}}$  and  $C$  is normal by [9, Proposition IV.2.1.13].  
(ii) The author used Zariski's subspace theorem in the proof of [13, (10)], but it derives only the inclusion  $\bar{\varphi}_*\mathcal{O}_{\bar{X}}(-l_2E_Z) \subset \mathcal{I}_Z^{(l)}$  for the  $l$ -th symbolic power  $\mathcal{I}_Z^{(l)}$  of  $\mathcal{I}_Z$ . In order to obtain [13, (10)], we need the equivalence of the  $\mathcal{I}_Z$ -adic topology and the  $\mathcal{I}_Z$ -symbolic topology by [35, §6 Lemma 3] (see also [26], [33]).

#### 4. THE ACC FOR MINIMAL LOG DISCREPANCIES

In this section, we discuss the ACC for minimal log discrepancies on non-singular varieties from the point of view of generic limits.

**4.A. Statements.** We begin with the statement of the ACC conjecture.

**Definition 4.1.** We say that a subset  $I$  of  $\mathbb{R}$  satisfies the *ascending chain condition (ACC)* (resp. the *descending chain condition (DCC)*) if there exist no infinite strictly increasing (resp. strictly decreasing) sequences of elements in  $I$ .

*Remark 4.2.*  $I \subset \mathbb{R}$  is finite if and only if  $I$  satisfies both the ACC and DCC.

**Definition 4.3.** Let  $P \in (X, \Delta = \sum_i \delta_i \Delta_i, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$  be a germ of a triplet. We write  $\text{Coef}_P(\Delta, \mathfrak{a})$  for the set which consists of all  $\delta_i > 0$  with  $\Delta_i$  passing through  $P$  and all  $r_j > 0$  with  $\mathfrak{a}_j$  non-trivial at  $P$ .

**Conjecture 4.4** (Shokurov [27], [29], McKernan [23]). *Fix  $d \in \mathbb{N}$  and subsets  $I \subset (0, \infty)$  and  $J \subset [0, \infty)$  both of which satisfy the DCC. Then there exist finite subsets  $I_0 \subset I$  and  $J_0 \subset J$  such that if  $P \in (X, \Delta, \mathfrak{a})$  is a germ of a triplet on a variety  $X$  of dimension  $d$  with  $\text{Coef}_P(\Delta, \mathfrak{a}) \subset I$  and  $\text{mld}_P(X, \Delta, \mathfrak{a}) \in J$ , then  $\text{Coef}_P(\Delta, \mathfrak{a}) \subset I_0$  and  $\text{mld}_P(X, \Delta, \mathfrak{a}) \in J_0$ .*

Conjecture 4.4 by McKernan is a generalisation of the original conjecture by Shokurov, which claims only the existence of  $J_0$ . When  $d = 2$ , the existence of  $J_0$  was proved by Alexeev [2]. The motivation of this conjecture stems from the reduction by Shokurov [30] that the termination of flips follows from two conjectural properties of minimal log discrepancies: the ACC and the lower semi-continuity. For the purpose of the termination of flips, one may assume  $I$  in Conjecture 4.4 to be a finite set.

We consider Conjecture 4.4 with the assumption of the non-singularity of  $X$ . Then we may assume  $\Delta = 0$  by absorbing  $\Delta$  to  $\mathfrak{a}$ , since any divisor on  $X$  is a Cartier divisor.

**Conjecture 4.4'.** Fix  $d \in \mathbb{N}$  and subsets  $I \subset (0, \infty)$  and  $J \subset [0, d]$  both of which satisfy the DCC. Then there exist finite subsets  $I_0 \subset I$  and  $J_0 \subset J$  such that if  $P \in (X, \mathfrak{a})$  is a germ of a pair on a non-singular variety  $X$  of dimension  $d$  with  $\text{Coef}_P \mathfrak{a} \subset I$  and  $\text{mld}_P(X, \mathfrak{a}) \in J$ , then  $\text{Coef}_P \mathfrak{a} \subset I_0$  and  $\text{mld}_P(X, \mathfrak{a}) \in J_0$ .

Theorem 1.3 is Conjecture 4.4' for  $d = 3$  with  $J \subset (1, 3]$ . Conjecture 4.4' with  $I$  finite was proved in [15].

**4.B. Reduction.** We shall reduce Conjecture 4.4' to the stability of minimal log discrepancies in taking a generic limit of  $\mathbb{R}$ -ideals. We refer to Appendix A for the definition of a generic limit and the relevant notation:  $R = k[[x_1, \dots, x_d]]$  with maximal ideal  $\mathfrak{m}$  and  $X = \text{Spec} R$  with closed point  $P$ , and for a field extension  $K$  of  $k$ ,  $R_K = K[[x_1, \dots, x_d]]$  with maximal ideal  $\mathfrak{m}_K$  and  $X_K = \text{Spec} R_K$  with closed point  $P_K$ .

**Conjecture 4.5** ([15, Conjecture 5.7]). Fix  $r_1, \dots, r_e > 0$ . Let  $S = \{(\mathfrak{a}_{i1}, \dots, \mathfrak{a}_{ie})\}_{i \in I}$  be a collection of  $e$ -tuples of ideals in  $R = k[[x_1, \dots, x_d]]$ , and  $(\mathfrak{a}_1, \dots, \mathfrak{a}_e)$  the generic limit of  $S$  defined in  $R_K$  with respect to a family  $\mathcal{F}$  of approximations of  $S$ . Set  $\mathfrak{a}_i = \prod_j \mathfrak{a}_{ij}^{r_{ij}}$  and  $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$ . Then after replacing  $\mathcal{F}$  with a subfamily,

$$\text{mld}_{P_K}(X_K, \mathfrak{a}) = \text{mld}_P(X, \mathfrak{a}_i)$$

for any  $i \in I$ .

Conjecture 4.5 is closely related to the ideal-adic semi-continuity of minimal log discrepancies.

**Conjecture 4.6** (Mustață, cf. [13, Conjecture 2.5]). Let  $P \in X = \text{Spec} k[[x_1, \dots, x_d]]$  and  $\mathfrak{m}$  be as above and  $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$  an  $\mathbb{R}$ -ideal on  $X$ . Then there exists an integer  $l$  such that if an  $\mathbb{R}$ -ideal  $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$  on  $X$  satisfies  $\mathfrak{a}_j + \mathfrak{m}^l = \mathfrak{b}_j + \mathfrak{m}^l$  for all  $j$ , then  $\text{mld}_P(X, \mathfrak{a}) = \text{mld}_P(X, \mathfrak{b})$ .

*Remark 4.7.* One inequality is easy in both conjectures. One has  $\text{mld}_{P_K}(X_K, \mathfrak{a}) \geq \text{mld}_P(X, \mathfrak{a}_i)$  in Conjecture 4.5 by Lemma A.8, and  $\text{mld}_P(X, \mathfrak{a}) \geq \text{mld}_P(X, \mathfrak{b})$  in Conjecture 4.6 by [13, Remark 2.5.3]. In particular, these conjectures hold in the case when  $(X_K, \mathfrak{a})$  (resp.  $(X, \mathfrak{a})$ ) is not lc.

**Proposition 4.8.** Conjecture 4.5 implies Conjectures 4.4' and 4.6.

*Proof.* Firstly, we shall see Conjecture 4.4'. It was observed by Mustață and sketched in [13, Remark 2.5.1]. Let  $\{\mathfrak{a}_i = \prod_{j=1}^{e_i} \mathfrak{a}_{ij}^{r_{ij}}\}_{i \in N}$  be an arbitrary collection of  $\mathbb{R}$ -ideals on  $X = \text{Spec} R$  such that  $\mathfrak{a}_{ij}$  are non-trivial at  $P$ ,  $r_{ij} \in I$  and  $m_i := \text{mld}_P(X, \mathfrak{a}_i) \in J$ . Then  $\sum_{j=1}^{e_i} r_{ij} \leq \text{ord}_E \mathfrak{a}_i \leq a_E(X) = d$  for the divisor  $E$  obtained by the blow-up of  $X$  at  $P$ , since  $m_i \geq 0$ . The  $I$  has the minimum, say  $\iota > 0$ , so  $e_i \leq \iota^{-1}d$ . By Corollary 2.3 and Remark 4.2, it is enough to show that both the subsets  $\bigcup_{i \in N} \text{Coef}_P \mathfrak{a}_i$  of  $I$  and  $\bigcup_{i \in N} \{m_i\}$  of  $J$  satisfy the ACC. We may replace  $N$  with a countable subset  $\mathbb{N}$  on which  $e_i$  is constant, say  $e$ , such that the sequences  $\{r_{ij}\}_{i \in \mathbb{N}}$  for  $1 \leq j \leq e$  and  $\{m_i\}_{i \in \mathbb{N}}$  are non-decreasing. By  $r_{ij} \leq d$  and  $m_i \leq d$ , these sequences have limits  $r_j := \lim_i r_{ij}$  and  $m := \lim_i m_i$ . It suffices to prove  $r_{ij} = r_j$  and  $m_i = m$  for some  $i$ .

For the collection  $S = \{(\mathfrak{a}_{i1}, \dots, \mathfrak{a}_{ie})\}_{i \in \mathbb{N}}$  of  $e$ -tuples of ideals in  $R$ , we take a family  $\mathcal{F} = (Z_l, (\bar{\mathfrak{a}}_j(l))_j, I_l, s_l, t_{l+1})_{l \geq l_0}$  of approximations of  $S$  and the generic limit  $(\mathfrak{a}_1, \dots, \mathfrak{a}_e)$  of  $S$  defined in  $R_K$  with respect to  $\mathcal{F}$  as in Lemma A.8, where

$E_K \in \mathcal{D}_{X_K}$  computing  $M := \text{mld}_{P_K}(X_K, \prod_j \mathfrak{a}_j^{r_j})$  is fixed. It is extended to  $E_l$  over  $X \times_{\text{Spec } k} Z_l$ , and for  $i \in I_l$  with  $z = s_l(i)$  we have  $M = \text{mld}_P(X, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}^l)^{r_j}) = a_{(E_l)_z}(X, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}^l)^{r_j})$  and  $\text{ord}_{E_K} \mathfrak{a}_j = \text{ord}_{(E_l)_z} (\mathfrak{a}_{ij} + \mathfrak{m}^l) < l$  using (iii) in Definition A.1. Hence  $\text{ord}_{E_K} \mathfrak{a}_j = \text{ord}_{(E_l)_z} \mathfrak{a}_{ij}$  and

$$(8) \quad m_i \leq a_{(E_l)_z}(X, \prod_j \mathfrak{a}_{ij}^{r_{ij}}) = a_{(E_l)_z}(X, \prod_j \mathfrak{a}_{ij}^{r_j}) + \sum_j (r_j - r_{ij}) \text{ord}_{(E_l)_z} \mathfrak{a}_{ij} \\ = M + \sum_j (r_j - r_{ij}) \text{ord}_{E_K} \mathfrak{a}_j.$$

By Conjecture 4.5,  $M = \text{mld}_P(X, \prod_j \mathfrak{a}_{ij}^{r_j}) \leq m_i$  for any  $i \in I_l$  after replacing  $\mathcal{F}$  with a subfamily. With (8), we obtain

$$M \leq m_i \leq M + \sum_j (r_j - r_{ij}) \text{ord}_{E_K} \mathfrak{a}_j.$$

The right-hand side converges to  $M$ , whence  $m_i = m = M$ . Then  $\text{mld}_P(X, \prod_j \mathfrak{a}_{ij}^{r_{ij}}) = \text{mld}_P(X, \prod_j \mathfrak{a}_{ij}^{r_j})$ , so  $r_{ij} = r_j$ .

Secondly, we shall see Conjecture 4.6. Suppose the contrary. Then for every  $i \in \mathbb{N}$ , there exists an  $\mathbb{R}$ -ideal  $\mathfrak{b}_i = \prod_j \mathfrak{b}_{ij}^{r_j}$  on  $X$  such that  $\mathfrak{a}_j + \mathfrak{m}^i = \mathfrak{b}_{ij} + \mathfrak{m}^i$  for all  $j$  but  $\text{mld}_P(X, \mathfrak{a}) \neq \text{mld}_P(X, \mathfrak{b}_i)$ . Take a family  $\mathcal{F} = (Z_l, (\bar{\mathfrak{b}}_j(l))_j, I_l, s_l, t_{l+1})_{l \geq l_0}$  of approximations of  $S = \{(\mathfrak{b}_{ij})_j\}_{i \in \mathbb{N}}$  and the generic limit  $(\mathfrak{b}_j)_j$  of  $S$  defined in  $R_K$  with respect to  $\mathcal{F}$ . Then for  $l \geq l_0$ ,  $\bar{\mathfrak{b}}_j(l)_z R = \mathfrak{b}_{ij} + \mathfrak{m}^l = \mathfrak{a}_j + \mathfrak{m}^l$  for  $i \in I_l$  with  $z = s_l(i)$  satisfying  $i \geq l$ , and such  $z$  form a dense subset of  $Z_l$ . This implies  $\bar{\mathfrak{b}}_j(l) = ((\mathfrak{a}_j + \mathfrak{m}^l) \cap \bar{R}) \otimes_k \mathcal{O}_{Z_l}$ , whence  $\bar{\mathfrak{b}}_j(l)_K = (\mathfrak{a}_j R_K + \mathfrak{m}_K^l) \cap \bar{R}_K$ . Then  $\mathfrak{b}_j = \varprojlim_l \bar{\mathfrak{b}}_j(l)_K = \mathfrak{a}_j R_K$  by Remark A.3, so  $\text{mld}_{P_K}(X_K, \prod_j \mathfrak{b}_j^{r_j}) = \text{mld}_P(X, \mathfrak{a})$  by Corollary 2.3. By Conjecture 4.5, we have  $\text{mld}_{P_K}(X_K, \prod_j \mathfrak{b}_j^{r_j}) = \text{mld}_P(X, \mathfrak{b}_i)$  for infinitely many  $i$ , that is,  $\text{mld}_P(X, \mathfrak{a}) = \text{mld}_P(X, \mathfrak{b}_i)$ , which is absurd. q.e.d.

*Remark 4.9.* Proposition 4.8 has the refinement that for fixed  $d$  and  $a \geq 0$ ,

- (i) Conjecture 4.5 for  $d$  with  $\text{mld}_{P_K}(X_K, \mathfrak{a}) > a$  (resp.  $\geq a$ ) implies Conjecture 4.4' for  $d$  with  $J \subset (a, d]$  (resp.  $\subset [a, d]$ ), and
- (ii) Conjecture 4.5 for  $d$  with  $\text{mld}_{P_K}(X_K, \mathfrak{a}) = a$  implies Conjecture 4.6 for  $d$  with  $\text{mld}_P(X, \mathfrak{a}) = a$ .

This is obvious by the above proof. Note that (8) implies  $m \leq M$ .

*Remark 4.10.* Theorem A.9 gives Conjecture 4.6 in the case when  $\text{mld}_P(X, \mathfrak{a}) = 0$ , and then its Corollary A.10 gives Conjecture 4.5 in the case when  $\text{mld}_{P_K}(X_K, \mathfrak{a}) = 0$ . The order of this logic is opposite to Proposition 4.8. We expect that an effective estimate of  $l$  in Conjecture 4.6 implies Conjecture 4.5.

Theorem A.9 is reduced to the corresponding statement [5, Theorem 1.4] on a variety by the property that the log canonical threshold for an ideal in  $\widehat{\mathcal{O}_{Y,Q}}$  is approximated by those for ideals in  $\mathcal{O}_{Y,Q}$ . This property for the minimal log discrepancy on  $X$  is a special case of Conjecture 4.5, so we do not know how to reduce Conjecture 4.6 to its variety version. The version of Conjecture 4.6 for a germ  $Q \in (Y, \Delta, \mathfrak{a})$  of a triplet on a variety  $Y$  holds when (i)  $(Y, \Delta, \mathfrak{a})$  is klt [13, Theorem 2.6], (ii)  $Y$  is a surface [14], or (iii)  $Y$  is toric and  $Q, \Delta, \mathfrak{a}$  are torus invariant [24, Theorem 1.8].

The variety version of Theorem A.9 is globalised.

**Theorem 4.11.** *Let  $(Y, \Delta, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$  be a triplet on a variety  $Y$  and  $Z$  an irreducible closed subset of  $Y$ . Suppose  $\text{mld}_{\eta_Z}(Y, \Delta, \mathfrak{a}) = 0$  and it is computed by  $E \in \mathcal{D}_Y$ . Then there exists an open subset  $Y'$  of  $Y$  containing  $\eta_Z$  such that if an  $\mathbb{R}$ -ideal  $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$  on  $Y'$  satisfies  $\mathfrak{a}_j|_{Y'} + \mathfrak{p}_j = \mathfrak{b}_j + \mathfrak{p}_j$  for all  $j$ , where  $\mathfrak{p}_j = \{u \in \mathcal{O}_{Y'} \mid \text{ord}_E u > \text{ord}_E \mathfrak{a}_j\}$ , then  $(Y', \Delta|_{Y'}, \mathfrak{b})$  is lc about  $Z|_{Y'}$  and  $\text{mld}_{\eta_Z}(Y', \Delta|_{Y'}, \mathfrak{b}) = 0$ .*

*Proof.* Take a log resolution  $f: W \rightarrow Y$  of  $(Y, \Delta, \mathfrak{a}\mathcal{I}_Z)$ , where  $\mathcal{I}_Z$  is the ideal sheaf of  $Z$ , such that  $E$  is realised as a divisor on  $W$ . Then  $F := \text{Exc } f \cup \text{Supp } \Delta_W \cup \text{Cosupp } \mathfrak{a}\mathcal{I}_Z\mathcal{O}_Y$  is an snc divisor  $\sum_i F_i$ , where  $\Delta_W$  is the strict transform of  $\Delta$ . By generic smoothness [10, Corollary III.10.7], there exists an open subset  $Y'$  of  $Y$  containing  $\xi_Z$  such that if the restriction  $S' = S|_{f^{-1}(Y')}$  of a stratum  $S$  of  $\sum_i F_i$  satisfies  $S' \neq \emptyset$  and  $f(S') \subset Z' = Z|_{Y'}$ , then  $S' \rightarrow Z'$  is smooth and surjective. Then for any  $Q \in Z'$ ,  $\text{mld}_Q(Y, \Delta, \mathfrak{a}\mathfrak{m}_Q^{\dim Z}) = 0$  for the maximal ideal sheaf  $\mathfrak{m}_Q$ , and it is computed by the divisor  $G_Q$  obtained by the blow-up of  $W$  along a component of  $E \cap f^{-1}(Q)$ . Since  $\text{ord}_{G_Q} \mathfrak{a}_j = \text{ord}_E \mathfrak{a}_j$  and  $\text{ord}_{G_Q} u \geq \text{ord}_E u$  for  $u \in \mathcal{O}_{Y'}$ , we have  $\text{mld}_Q(Y', \Delta|_{Y'}, \mathfrak{b}\mathfrak{m}_Q^{\dim Z}) = 0$  for  $\mathfrak{b}$  in Theorem 4.11 by [5, Theorem 1.4] (its proof works for triplets). Hence  $(Y', \Delta|_{Y'}, \mathfrak{b})$  is lc about  $Z'$ , and  $\text{mld}_{\eta_Z}(Y', \Delta|_{Y'}, \mathfrak{b}) = 0$  by  $a_E(Y', \Delta|_{Y'}, \mathfrak{b}) = 0$ . q.e.d.

**Corollary 4.12.** *Let  $(Y, \Delta, \mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j})$  be an lc triplet on a variety  $Y$  and  $Z$  a closed subset of  $Y$  with ideal sheaf  $\mathcal{I}_Z$ . Then there exists an integer  $l$  such that if an  $\mathbb{R}$ -ideal  $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$  on  $Y$  satisfies  $\mathfrak{a}_j + \mathcal{I}_Z^l = \mathfrak{b}_j + \mathcal{I}_Z^l$  for all  $j$ , then  $(Y, \Delta, \mathfrak{b})$  is lc about  $Z$ .*

*Remark 4.13.* The author should have written the proof after [13, Theorem 2.4]. The estimate of  $l$  in [13, Remark 2.4.1] is an error unless  $Z$  is a closed point (so is that of  $l_1$  in [13, Lemma 3.1]).

## 5. THE KLT AND PLT CASES

In this section, we settle Conjecture 4.5 in the klt case, and in the plt case whose lc centre has an isolated singularity. We keep the notation in Appendix A, so  $P \in X = \text{Spec } R$  with  $R = k[[x_1, \dots, x_d]]$  and  $P_K \in X_K = \text{Spec } R_K$  with  $R_K = K[[x_1, \dots, x_d]]$ .

### 5.A. The klt case.

**Theorem 5.1.** *Conjecture 4.5 holds in the case when  $(X_K, \mathfrak{a})$  is klt.*

*Proof.* It is shown similarly to [13, Theorem 2.6]. By Remark 4.7, it suffices to show that after replacing  $\mathcal{F}$  with a subfamily,

$$(9) \quad a_E(X, \mathfrak{a}_i) \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$$

for any  $i \in I_l$  and  $E \in \mathcal{D}_X$  with centre  $P$ .

Take a subfamily in Lemma A.8 so that  $\text{mld}_P(X, \prod_j (\bar{\mathfrak{a}}_j(l)_z R)^{r_j}) = \text{mld}_{P_K}(X_K, \mathfrak{a})$  for  $z \in Z_l$ . Then for  $i \in I_l$ ,

$$(10) \quad a_E(X, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}^l)^{r_j}) \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$$

by (iii) in Definition A.1. Since  $(X_K, \mathfrak{a})$  is klt, we can fix  $t > 0$  such that  $(X_K, \mathfrak{a}^{1+t})$  is lc. By Corollary A.10,  $(X, \mathfrak{a}_i^{1+t})$  is lc for  $i \in I_l$  after replacing  $\mathcal{F}$  with a subfamily, whence  $a_E(X, \mathfrak{a}_i) \geq t \text{ord}_E \mathfrak{a}_i = t \sum_j r_j \text{ord}_E \mathfrak{a}_{ij}$ . We fix  $l \geq l_0$  such that

$l \geq (tr_j)^{-1} \text{mld}_{P_K}(X_K, \mathfrak{a})$  for all  $j$ . Then,

$$(11) \quad a_E(X, \mathfrak{a}_i) \geq l^{-1} \text{ord}_E \mathfrak{a}_{ij} \cdot \text{mld}_{P_K}(X_K, \mathfrak{a})$$

for any  $j$  and  $i \in I_l$ .

If  $\text{ord}_E \mathfrak{a}_{ij} < l$  for all  $j$ , then  $\text{ord}_E \mathfrak{a}_{ij} = \text{ord}_E(\mathfrak{a}_{ij} + \mathfrak{m}^l)$ , so one has  $a_E(X, \mathfrak{a}_i) = a_E(X, \prod_j (\mathfrak{a}_{ij} + \mathfrak{m}^l)^{r_j})$ , and (9) follows from (10). If  $\text{ord}_E \mathfrak{a}_{ij} \geq l$  for some  $j$ , then (9) follows from (11). q.e.d.

*Remark 5.2.* By Remark 4.7, Theorem 5.1 and Corollary A.10, Conjecture 4.5 remains open only when  $(X_K, \mathfrak{a})$  is non-klt with  $\text{mld}_{P_K}(X_K, \mathfrak{a}) > 0$ .

**5.B. The plt case whose lc centre has an isolated singularity.** Suppose that  $(X_K, \mathfrak{a})$  is an lc but not klt pair every lc centre of which has codimension 1. Then by Proposition 3.7,  $(X_K, \mathfrak{a})$  has the smallest lc centre  $S_K$  and it is normal. We prove Conjecture 4.5 on the assumption that  $S_K$  has an isolated singularity.

**Theorem 5.3.** *Conjecture 4.5 holds in the case when  $(X_K, \mathfrak{a})$  has the smallest lc centre of codimension 1 which is non-singular outside  $P_K$ .*

We let  $S_K$  denote the smallest lc centre of  $(X_K, \mathfrak{a})$ .  $S_K$  is a prime divisor which is non-singular outside  $P_K$ . We define an  $\mathbb{R}$ -ideal  $\mathfrak{c} = \prod_j \mathfrak{c}_j^{r_j}$  by the expression  $\mathfrak{a}_j = \mathfrak{c}_j \mathcal{O}_{X_K}(-m_j S_K)$  with  $\sum_j r_j m_j = 1$ . The  $\mathfrak{a}$  and  $\mathfrak{c} \mathcal{O}_{X_K}(-S_K)$  take the same order along any divisor over  $X_K$ . We can fix  $t > 0$  such that  $(X_K, S_K, \mathfrak{c}^{1+t})$  is lc, since  $S_K$  is the unique lc centre of  $(X_K, S_K, \mathfrak{c})$ .

We take a log resolution  $f_K: Y_K \rightarrow X_K$  of  $(X_K, S_K, \mathfrak{m}_K)$ , which is isomorphic outside  $P_K$ . Let  $\{E_{\alpha K}\}_\alpha$  be the set of all  $f_K$ -exceptional prime divisors. The  $E_K = \sum_\alpha E_{\alpha K}$  is snc. Let  $(Y_K, \Delta_K, \mathfrak{a}' = \prod_j (\mathfrak{a}'_j)^{r_j})$  be the pull-back of  $(X_K, 0, \mathfrak{a})$  and  $(Y_K, T_K + \Delta_K, \mathfrak{c}')$  that of  $(X_K, S_K, \mathfrak{c})$ . We set

$$\begin{aligned} L_K &:= T_K \cap f_K^{-1}(P_K), \\ C_K &:= \text{Cosupp } \mathfrak{c}' \cap f_K^{-1}(P_K). \end{aligned}$$

By blowing up  $Y_K$  further, we may assume that  $C_K$  is contained in the union of those  $E_{\alpha K}$  satisfying

$$(12) \quad t \text{ord}_{E_{\alpha K}} \mathfrak{c} \geq \text{mld}_{P_K}(X_K, \mathfrak{a}).$$

One sees this by induction on  $\max_J \{\min_{\alpha \in J} \{\text{ord}_{E_{\alpha K}} \mathfrak{c}\}\}$  in which one considers all subsets  $J$  of indices satisfying  $C_K \subset \bigcup_{\alpha \in J} E_{\alpha K}$ , since the order of  $\mathfrak{c}$  takes value in the discrete subset  $\sum_j r_j \mathbb{Z}_{\geq 0}$  of  $\mathbb{R}$ .

The  $f_K$  is descendible by Proposition A.7, so replacing  $\mathcal{F}$  with a subfamily, we obtain the diagram (15) in which  $\tilde{f}_l$  is a family of log resolutions. Shrinking  $Z_l$ , we may assume that  $E_{\alpha K}$ ,  $L_K$  and  $C_K$  are the base changes of flat families  $\tilde{E}_{\alpha l}$ ,  $\tilde{L}_l$  and  $\tilde{C}_l$  in  $\tilde{Y}_l$  over  $Z_l$ . We may assume that  $\sum_\alpha \tilde{E}_{\alpha l}$  is an snc divisor, that the projections to  $Z_l$  from every stratum of  $\sum_\alpha \tilde{E}_{\alpha l}$  and from its intersection with  $\tilde{L}_l$  are smooth and surjective, and that  $\text{ord}_{(\tilde{E}_{\alpha l})_z} \tilde{\mathfrak{a}}_j(l)_z$  is constant on  $z \in Z_l$  for each  $\alpha$  and  $j$ . Their base changes in  $Y_l$  are denoted by  $E_{\alpha l}$ ,  $L_l$  and  $C_l$ . We write  $\tilde{E}_l = \sum_\alpha \tilde{E}_{\alpha l}$  and  $E_l = \sum_\alpha E_{\alpha l}$ .

We fix  $m$  such that  $m \text{ord}_{E_{\alpha K}} \mathfrak{m}_K \geq \text{ord}_{E_{\alpha K}} \mathfrak{c}_j$  for all  $\alpha$  and  $j$ , and set

$$\mathfrak{d} := \prod_j (\mathfrak{c}_j + \mathfrak{m}_K^m)^{tr_j}.$$

Then  $\text{ord}_{E_{\alpha K}} \mathfrak{d} = t \text{ord}_{E_{\alpha K}} \mathfrak{c}$ , and  $(X_K, \mathfrak{a}\mathfrak{d})$  is lc. The  $\mathfrak{d}$  is defined over some  $k(Z_l)$ , so by replacing  $\mathcal{F}$  with a subfamily, we may assume that  $\mathfrak{d}$  is the base change of an

$\mathbb{R}$ -ideal  $\bar{\mathfrak{d}}_l = \prod_j \bar{\mathfrak{d}}_{l_j}^{tr_j}$  on  $\mathbb{A}_k^d \times_{\text{Spec } k} Z_l$  with  $\bar{\mathfrak{m}}^m \otimes_k \mathcal{O}_{Z_l} \subset \bar{\mathfrak{d}}_{l_j}$  and that  $\text{ord}_{(\bar{E}_{\alpha l})_z}(\bar{\mathfrak{d}}_l)_z$  is constant on  $Z_l$  for each  $\alpha$ . By Corollary A.10, after taking a subfamily,  $(X, \mathfrak{a}_i(\mathfrak{d}_l)_z)$  is lc for any  $i \in I_l$  with  $z = s_l(i)$ , where  $\mathfrak{d}_l$  is the pull-back on  $X \times_{\text{Spec } k} Z_l$  of  $\bar{\mathfrak{d}}_l$ .

We fix  $l \geq l_0$  such that

$$(13) \quad l \text{ord}_{E_{\alpha K}} \mathfrak{m}_K > \text{ord}_{E_{\alpha K}} \mathfrak{a}_j + \text{ord}_{S_K} \mathfrak{a}_j$$

for all  $\alpha$  and  $j$ . By Remark 4.7, for Theorem 5.3 it suffices to prove that after shrinking  $Z_l$ ,

$$(14) \quad a_E(X, \mathfrak{a}_i) \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$$

for any  $i \in I_l$  and  $E \in \mathcal{D}_X$  with centre  $P$ . Setting  $z = s_l(i)$ , we shall prove (14) by treating the three cases according to the position of  $c_{(Y_l)_z}(E)$ :

- (a)  $c_{(Y_l)_z}(E) \not\subset (L_l \cup C_l)_z$ .
- (b)  $c_{(Y_l)_z}(E) \subset (C_l)_z$ .
- (c)  $c_{(Y_l)_z}(E) \subset (L_l)_z$  and  $c_{(Y_l)_z}(E) \not\subset (C_l)_z$ .

We let  $\bar{\mathfrak{a}}'(l) = \prod_j \bar{\mathfrak{a}}'_j(l)^{r_j}$  be the weak transform on  $\bar{Y}_l$  of  $\prod_j \bar{\mathfrak{a}}_j(l)^{r_j}$ , and  $\mathfrak{a}'_i = \prod_j (\mathfrak{a}'_{ij})^{r_j}$  the weak transform on  $(Y_l)_z$  of  $\mathfrak{a}_i$ .

**Lemma 5.4.** (i)  $\bar{\mathfrak{a}}'_j(l)_K \mathcal{O}_{Y_K} = \mathfrak{a}'_j + l_{lj}$  with an ideal sheaf  $l_{lj}$  which is contained in  $\mathcal{O}_{Y_K}(-E_K)^{\text{ord}_{S_K} \mathfrak{a}_j + 1}$ .  
(ii)  $\bar{\mathfrak{a}}'_j(l)_z \mathcal{O}_{(Y_l)_z} = \mathfrak{a}'_{ij} + \mathcal{I}_{lij}$  with an ideal sheaf  $\mathcal{I}_{lij}$  which is contained in  $\mathcal{O}_{(Y_l)_z}(-(E_l)_z)^{\text{ord}_{S_K} \mathfrak{a}_j + 1}$ .  
(iii)  $\text{Cosupp } \bar{\mathfrak{a}}'(l) = \bar{L}_l \cup \bar{C}_l$  after shrinking  $Z_l$ .

*Proof.* Write  $\mathfrak{m}_K \mathcal{O}_{Y_K} = \mathcal{O}_{Y_K}(-M_K)$  and  $\mathfrak{a}_j \mathcal{O}_{Y_K} = \mathfrak{a}'_j \mathcal{O}_{Y_K}(-A_{jK})$ . The inequality (13) means that  $l_{lj} = \mathcal{O}_{Y_K}(A_{jK} - lM_K)$  is an ideal sheaf contained in  $\mathcal{O}_{Y_K}(-E_K)^{\text{ord}_{S_K} \mathfrak{a}_j + 1}$ . Then  $\mathfrak{m}_K^l \mathcal{O}_{Y_K} = l_{lj} \mathcal{O}_{Y_K}(-A_{jK})$ , which induces (i) by  $\bar{\mathfrak{a}}_j(l)_K R_K = \mathfrak{a}_j + \mathfrak{m}_K^l$ . From (i),  $\text{Cosupp}(\bar{\mathfrak{a}}'(l)_K \mathcal{O}_{Y_K}) = \text{Cosupp } \mathfrak{a}' \cap f_K^{-1}(P_K) = L_K \cup C_K$ , which is extended to  $\text{Cosupp } \bar{\mathfrak{a}}'(l) = \bar{L}_l \cup \bar{C}_l$  in (iii). On the other hand,  $\text{ord}_{E_{\alpha K}} \mathfrak{m}_K = \text{ord}_{(E_{\alpha l})_z} \mathfrak{m}$  and  $\text{ord}_{E_{\alpha K}} \mathfrak{a}_j = \text{ord}_{E_{\alpha K}} \bar{\mathfrak{a}}_j(l)_K R_K = \text{ord}_{(\bar{E}_{\alpha l})_z} \bar{\mathfrak{a}}_j(l)_z = \text{ord}_{(E_{\alpha l})_z} \mathfrak{a}_{ij}$  by (13) and Definitions A.1, A.2. Then, (ii) is induced similarly to (i). q.e.d.

The cases (a) and (b) are not difficult.

*Proof of (14) in the case (a).* Set  $\Delta_l = \sum_{\alpha} (1 - a_{E_{\alpha K}}(X_K, \mathfrak{a})) E_{\alpha l}$ , base-changed to  $\Delta_K$ . Then  $((Y_l)_z, (\Delta_l)_z, \mathfrak{a}'_i)$  is the pull-back of  $(X, 0, \mathfrak{a}_i)$ . We have  $a_E((Y_l)_z, (\Delta_l)_z) \geq \text{ord}_E(E_l - \Delta_l)_z$  by the log canonicity of  $((Y_l)_z, (E_l)_z)$ . For a divisor  $(E_{\alpha l})_z$  containing  $c_{(Y_l)_z}(E)$ , we have  $\text{ord}_E(E_l - \Delta_l)_z \geq \text{ord}_{(E_{\alpha l})_z}(E_l - \Delta_l)_z = a_{E_{\alpha K}}(X_K, \mathfrak{a}) \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$ . Hence  $a_E((Y_l)_z, (\Delta_l)_z) \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$ . By Lemma 5.4(ii) and (iii),  $\text{Cosupp } \mathfrak{a}'_i \cap (f_l)_z^{-1}(P) = \text{Cosupp } \bar{\mathfrak{a}}'(l)_z \mathcal{O}_{(Y_l)_z} = (L_l \cup C_l)_z$ , so  $\text{ord}_E \mathfrak{a}'_i = 0$ . Thus  $a_E(X, \mathfrak{a}_i) = a_E((Y_l)_z, (\Delta_l)_z, \mathfrak{a}'_i) = a_E((Y_l)_z, (\Delta_l)_z) \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$ . q.e.d.

*Proof of (14) in the case (b).* The  $c_{(Y_l)_z}(E)$  lies on some  $(E_{\alpha l})_z$  such that  $E_{\alpha K}$  satisfies (12). Then  $\text{ord}_E(\mathfrak{d}_l)_z \geq \text{ord}_{(E_{\alpha l})_z}(\mathfrak{d}_l)_z = \text{ord}_{E_{\alpha K}} \mathfrak{d} = t \text{ord}_{E_{\alpha K}} \mathfrak{c} \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$ . On the other hand, the log canonicity of  $(X, \mathfrak{a}_i(\mathfrak{d}_l)_z)$  implies  $a_E(X, \mathfrak{a}_i) \geq \text{ord}_E(\mathfrak{d}_l)_z$ . These two inequalities are joined as  $a_E(X, \mathfrak{a}_i) \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$ . q.e.d.

The case (c) is reduced to the following log canonicity.

**Lemma 5.5.** *After shrinking  $Z_l$ , the triplet  $((Y_l)_z, (E_l)_z, \mathfrak{a}'_i)$  is lc about  $(L_l)_z \setminus (C_l)_z$  for any  $i \in I_l$  with  $z = s_l(i)$ .*

*Proof of (14) in the case (c) from Lemma 5.5.* For  $\Delta_l = \sum_{\alpha}(1 - a_{E_{\alpha K}}(X_K, \mathfrak{a}))E_{\alpha l}$ , we have  $a_E(X, \mathfrak{a}_i) = a_E((Y_l)_z, (\Delta_l)_z, \mathfrak{a}'_i) \geq \text{ord}_E(E_l - \Delta_l)_z$  by Lemma 5.5, and have seen  $\text{ord}_E(E_l - \Delta_l)_z \geq \text{mld}_{P_K}(X_K, \mathfrak{a})$  in the proof in the case (a). q.e.d.

*Proof of Lemma 5.5.* Pick any open stratum  $F_K$  of the snc divisor  $E_K$ , which is extended to an open stratum  $\bar{F}_l$  of  $\bar{E}_l$ . We prove Lemma 5.5 by noetherian induction. Recall that  $l$  has been fixed. Let  $\bar{Q}_l$  be an irreducible locally closed subset of  $\bar{F}_l \cap \bar{L}_l \setminus \bar{C}_l$  which dominates  $Z_l$ . It suffices to show that the existence of a dense open subset  $\bar{Q}_l^\circ$  of  $\bar{Q}_l$  such that the triplet  $((Y_l)_z, (E_l)_z, \mathfrak{a}'_i)$  is lc about  $(Q_l^\circ)_z$  for  $i \in I_l$  with  $z = s_l(i)$ , where  $Q_l^\circ = \bar{Q}_l^\circ \times_{\bar{Y}_l} Y_l$ .

By shrinking  $\bar{Q}_l$  and  $Z_l$ , we may assume that  $\bar{Q}_l \rightarrow Z_l$  is smooth and surjective. Let  $\bar{f}_l^+ : \bar{Y}_l^+ \rightarrow \mathbb{A}_k^d \times_{\text{Spec } k} \bar{Q}_l$  be the base change of  $\bar{f}_l$  by  $\bar{Q}_l \rightarrow Z_l$ . Then  $\text{pr}_{\bar{Q}_l} \circ \bar{f}_l^+$  has the natural section  $\bar{g}_l : \bar{Q}_l \rightarrow \bar{Y}_l^+ = \bar{Y}_l \times_{Z_l} \bar{Q}_l$  by the immersion  $\bar{Q}_l \hookrightarrow \bar{Y}_l$ . We construct  $f_K^+$  and  $g_K$  similarly for  $Q_K = \bar{Q}_l \times_{\bar{Y}_l} Y_K$  as below.

$$\begin{array}{ccccc}
 Y_K^+ & \xrightarrow{\quad} & \bar{Y}_l^+ & \xrightarrow{\quad} & \bar{Y}_l \\
 \downarrow f_K^+ & & \downarrow \bar{f}_l^+ & & \downarrow \bar{f}_l \\
 X_K \times_{\text{Spec } K} Q_K & \xrightarrow{\quad} & \mathbb{A}_k^d \times_{\text{Spec } k} \bar{Q}_l & \xrightarrow{\quad} & \mathbb{A}_k^d \times_{\text{Spec } k} Z_l \\
 \downarrow & & \downarrow & & \downarrow \\
 Q_K & \xrightarrow{\quad} & \bar{Q}_l & \xrightarrow{\quad} & Z_l
 \end{array}$$

$g_K$  (curved arrow from  $Q_K$  to  $Y_K^+$ )       $\bar{g}_l$  (curved arrow from  $\bar{Q}_l$  to  $\bar{Y}_l^+$ )

The  $\bar{Y}_l^+, Y_K^+$  are the base changes of  $\bar{Y}_l, Y_K$  by smooth morphisms. For a staff  $\square$  on  $\bar{Y}_l$  or  $Y_K$ , we mean by  $\square^+$  the base change of  $\square$  on  $\bar{Y}_l^+$  or  $Y_K^+$ . For example,  $\mathfrak{a}'^+ = \prod_j (\mathfrak{a}'_j)^{r_j} = \mathfrak{a}' \mathcal{O}_{Y_K^+}$ . Let  $\bar{q}_l$  be the ideal sheaf of  $\bar{g}_l(\bar{Q}_l)$  on  $\bar{Y}_l^+$  and  $\bar{G}_l \in \mathcal{D}_{\bar{Y}_l^+}$  the divisor obtained by the blow-up of  $\bar{Y}_l^+$  along  $\bar{q}_l$ . They are base-changed to  $q$  on  $Y_K^+$  and  $G_K \in \mathcal{D}_{Y_K^+}$ .

We see that  $\text{mld}_{\eta_{g_K(Q_K)}}(Y_K^+, E_K^+, q^n \mathfrak{a}'^+) = 0$  with  $n = \dim F_K - 1$  and it is computed by  $G_K$ . We have  $\bar{\mathfrak{a}}'(l)_K^+ \mathcal{O}_{Y_K^+} = \prod_j (\mathfrak{a}'_j + l_{l_j}^+)^{r_j}$  and  $\text{ord}_{G_K} \mathfrak{a}'_j^+ = \text{ord}_{S_K} \mathfrak{a}_j < \text{ord}_{G_K} l_{l_j}^+$  from Lemma 5.4(i), so  $\text{mld}_{\eta_{g_K(Q_K)}}(Y_K^+, E_K^+, q^n \bar{\mathfrak{a}}'(l)_K^+ \mathcal{O}_{Y_K^+}) = 0$  and it is computed by  $G_K$ . Then  $\text{mld}_{\eta_{\bar{g}_l(\bar{Q}_l)}}(\bar{Y}_l^+, \bar{E}_l^+, \bar{q}_l^n \bar{\mathfrak{a}}'(l)^+) = 0$  and it is computed by  $\bar{G}_l$ . We regard  $\bar{Y}_l^+$  as a family over  $\bar{Q}_l$ . There exists a dense open subset  $\bar{Q}_l^\circ$  of  $\bar{Q}_l$  such that for any closed point  $q \in Q_l^\circ = \bar{Q}_l^\circ \times_{\bar{Y}_l} Y_l$  with its image  $z \in Z_l$ ,  $\text{mld}_q((Y_l)_z, (E_l)_z, \mathfrak{m}_q^n \bar{\mathfrak{a}}'(l) \mathcal{O}_{(Y_l)_z}) = 0$ , computed by  $(G_l)_q$ , and  $\text{ord}_{(G_l)_q} \bar{\mathfrak{a}}'_j(l) \mathcal{O}_{(Y_l)_z} = \text{ord}_{S_K} \mathfrak{a}_j$ , where  $\mathfrak{m}_q$  is the maximal ideal sheaf of  $q \in (Y_l)_z$  and  $G_l = \bar{G}_l \times_{\bar{Y}_l} Y_l$ . The  $(G_l)_q$  is obtained by the blow-up of  $(Y_l)_z$  at  $q$ . For  $i \in I_l$  with  $z = s_l(i)$ ,  $\bar{\mathfrak{a}}'(l) \mathcal{O}_{(Y_l)_z} = \prod_j (\mathfrak{a}'_{ij} + \mathcal{I}_{lij})^{r_j}$  and  $\text{ord}_{(G_l)_q} \bar{\mathfrak{a}}'_j(l) \mathcal{O}_{(Y_l)_z} < \text{ord}_{(G_l)_q} \mathcal{I}_{lij}$  by Lemma 5.4(ii). Applying Theorem A.9, we have  $\text{mld}_q((Y_l)_z, (E_l)_z, \mathfrak{m}_q^n \mathfrak{a}'_i) = 0$ , and the log canonicity of  $((Y_l)_z, (E_l)_z, \mathfrak{a}'_i)$  about  $(Q_l^\circ)_z$  is concluded. q.e.d.

Theorem 5.3 is completed.

## 6. THE THREEFOLD CASE

We shall prove Theorem 1.3. By Remark 4.9, the theorem follows from Conjecture 4.5 for  $d = 3$  with  $\text{mld}_{P_K}(X_K, \mathfrak{a}) > 1$ . In Remark 5.2, Conjecture 4.5 is reduced to the case when  $(X_K, \mathfrak{a})$  is an lc pair which has a minimal lc centre  $Z$  of positive

dimension. If  $d = 3$ , then by Theorem 1.2,  $Z$  is the smallest lc centre and it is normal. If  $Z$  is a surface, then one can apply Theorem 5.3. If  $Z$  is a curve, then  $\text{mld}_{P_K}(X_K, \mathfrak{a}) \leq 1$  by Proposition 6.1. Therefore, we obtain Theorem 1.3.

**Proposition 6.1.** *Let  $P \in (X, \mathfrak{a})$  be a germ of an lc pair on a non-singular  $R$ -variety  $X$  of dimension 3 with  $R = K[[x_1, \dots, x_d]]$  whose smallest lc centre is a curve. Then  $\text{mld}_P(X, \mathfrak{a}) \leq 1$ .*

*Proof.* The smallest lc centre  $C$  of  $(X, \mathfrak{a})$  is non-singular by Theorem 1.2. Setting  $(X_0, \Delta_0, \mathfrak{a}_0) := (X, 0, \mathfrak{a})$  and  $C_0 := C$ , we build a tower of finitely many blow-ups

$$X_n \rightarrow \cdots \rightarrow X_i \xrightarrow{f_i} X_{i-1} \rightarrow \cdots \rightarrow X_0 = X$$

such that (i)  $f_i: X_i \rightarrow X_{i-1}$  is the blow-up along  $C_{i-1}$ , (ii)  $E_i$  is the exceptional divisor of  $f_i$ , (iii)  $(X_i, \Delta_i, \mathfrak{a}_i)$  is the pull-back of  $(X_{i-1}, \Delta_{i-1}, \mathfrak{a}_{i-1})$ , (iv)  $C_i$  is a non-singular non-klt centre on  $X_i$  of  $(X, \mathfrak{a})$  mapped onto  $C_{i-1}$ , and (v)  $a_{E_i}(X, \mathfrak{a}) > 0$  for  $i < n$  and  $a_{E_n}(X, \mathfrak{a}) = 0$ . Here one can prove the effectiveness  $\Delta_i \geq 0$  and the non-singularity of  $C_i$  by induction. Indeed, if they hold for  $i - 1$ , then  $\text{ord}_{E_i} \Delta_i = \text{ord}_{C_{i-1}} \Delta_{i-1} + \text{ord}_{C_{i-1}} \mathfrak{a}_{i-1} - 1 > 0$  by Lemma 6.2. Unless  $a_{E_i}(X, \mathfrak{a}) = 0$ , an arbitrary lc centre  $C_i$  of  $(X_i, \Delta_i, \mathfrak{a}_i)$  mapped onto  $C_{i-1}$  is a curve and is minimal. The non-singularity of  $C_i$  follows from Theorem 1.2.

Let  $F$  be the divisor obtained by the blow-up of  $X_n$  along a curve in  $E_n \cap (f_1 \circ \cdots \circ f_n)^{-1}(P)$ . Then  $a_F(X, \mathfrak{a}) = a_F(X_n, \Delta_n, \mathfrak{a}_n) \leq a_F(X_n, E_n) = 1$  by  $\Delta_n \geq (1 - a_{E_n}(X, \mathfrak{a}))E_n = E_n$ . q.e.d.

**Lemma 6.2.** *Let  $(X, \mathfrak{a})$  be a pair on a non-singular  $R$ -variety  $X$  and  $Z$  a non-klt centre of  $(X, \mathfrak{a})$ . Then  $\text{ord}_Z \mathfrak{a} \geq 1$ . If in addition  $\text{codim}_X Z \geq 2$ , then  $\text{ord}_Z \mathfrak{a} > 1$ .*

*Proof.* The lemma is obvious if  $Z$  is a divisor, so we may assume  $\text{codim}_X Z \geq 2$ . Setting  $X_0 := X$ ,  $Z_0 := Z$  and  $\mathfrak{a}_0 := \mathfrak{a}$ , we build a tower of finitely many blow-ups

$$X_n \rightarrow \cdots \rightarrow X_i \xrightarrow{f_i} X_{i-1} \rightarrow \cdots \rightarrow X_0 = X$$

such that (i)  $f_i$  is the composition  $X_i \xrightarrow{h_i} X'_{i-1} \xrightarrow{g_{i-1}} X_{i-1}$  of the blow-up  $h_i: X_i \rightarrow X'_{i-1}$  along the strict transform  $Z'_{i-1}$  on  $X'_{i-1}$  of  $Z_{i-1}$  and an embedded resolution  $g_{i-1}: X'_{i-1} \rightarrow X_{i-1}$  of singularities of  $Z_{i-1}$ , in which  $g_{i-1}$  is isomorphic outside the singular locus of  $Z_{i-1}$ , (ii)  $E_i$  is the exceptional divisor of  $h_i$ , (iii)  $\mathfrak{a}_i$  is the weak transform on  $X_i$  of  $\mathfrak{a}_{i-1}$ , (iv)  $Z_i$  is a non-klt centre on  $X_i$  of  $(X, \mathfrak{a})$  mapped onto  $Z_{i-1}$ , and (v)  $a_{E_i}(X, \mathfrak{a}) > 0$  for  $i < n$  and  $a_{E_n}(X, \mathfrak{a}) \leq 0$ .

Supposing  $\text{ord}_Z \mathfrak{a} \leq 1$ , we shall derive by induction two inequalities  $\text{ord}_{Z_i} \mathfrak{a}_i \leq 1$  and  $a_{E_n}(X_i, \mathfrak{a}_i) \leq 0$  for any  $i$ . The claim for  $i = 0$  is trivial. If they hold for  $i - 1$ , then  $\text{ord}_{Z_i} \mathfrak{a}_i \leq \text{ord}_{V_i} \mathfrak{a}_i \leq \text{ord}_{Z_{i-1}} \mathfrak{a}_{i-1} \leq 1$  by [11, Lemmata III.7, III.8] for an irreducible closed subset  $V_i$  of  $Z_i$  meeting the non-singular locus of  $Z_i$  such that  $V_i \rightarrow Z_{i-1}$  is finite and surjective. Note that the symbol  $v^{(1)}$  in [11] stands for the order. The triplet  $(X_{i-1}, 0, \mathfrak{a}_{i-1})$  is pulled back to  $(X_i, \Delta_i, \mathfrak{a}_i)$  with  $\text{ord}_{E_i} \Delta_i = 1 + \text{ord}_{Z_{i-1}} \mathfrak{a}_{i-1} - \text{codim}_{X_{i-1}} Z_{i-1} \leq 0$ , so  $a_{E_n}(X_i, \mathfrak{a}_i) \leq a_{E_n}(X_i, \Delta_i, \mathfrak{a}_i) = a_{E_n}(X_{i-1}, \mathfrak{a}_{i-1}) \leq 0$ .

We obtained  $a_{E_n}(X_n, \mathfrak{a}_n) \leq 0$ . However, it contradicts  $a_{E_n}(X_n) = 1$  and  $\text{ord}_{E_n} \mathfrak{a}_n = 0$ . q.e.d.

## APPENDIX A. GENERIC LIMITS

The generic limit is a limit of ideals. It was constructed first by de Fernex and Mustařă [7] using ultraproducts, and then by Kollár [19] using Hilbert schemes.



We set  $\bar{R} = k[x_1, \dots, x_d]$  with maximal ideal  $\bar{\mathfrak{m}}$ , and  $\mathbb{A}_k^d = \text{Spec } \bar{R}$  with origin  $\bar{P}$ . We also set  $R = k[[x_1, \dots, x_d]]$  with  $\mathfrak{m} = \bar{\mathfrak{m}}R$ , and  $X = \text{Spec } R$  with closed point  $P$ . Mostly we discuss on the spectrum of a noetherian ring, where an ideal in the ring is identified with its coherent ideal sheaf.

We introduce the notion of a family of approximated ideals by which a generic limit is defined.

**Definition A.1.** Let  $S = \{(\mathfrak{a}_{i1}, \dots, \mathfrak{a}_{ie})\}_{i \in I}$  be a collection of  $e$ -tuples of ideals in  $R$ , indexed by an infinite set  $I$ . A family  $\mathcal{F}$  of approximations of  $S$  consists of, with  $l_0$  fixed, for each  $l \geq l_0$ ,

- (a) a variety  $Z_l$ ,
- (b) an ideal sheaf  $\bar{\mathfrak{a}}_j(l)$  on  $\mathbb{A}_k^d \times_{\text{Spec } k} Z_l$  containing  $\bar{\mathfrak{m}}^l \otimes_k \mathcal{O}_{Z_l}$  for  $1 \leq j \leq e$ ,
- (c) an infinite subset  $I_l$  of  $I$  and a map  $s_l: I_l \rightarrow Z_l(k)$ , where  $Z_l(k)$  is the set of  $k$ -points on  $Z_l$ , and
- (d) a dominant morphism  $t_{l+1}: Z_{l+1} \rightarrow Z_l$ ,

such that

- (i)  $\bar{\mathfrak{a}}_j(l)$  gives a flat family of closed subschemes of  $\mathbb{A}_k^d$  parametrised by  $Z_l$ ,
- (ii) the pull-back of  $\bar{\mathfrak{a}}_j(l)$  by  $\text{id}_{\mathbb{A}_k^d} \times t_{l+1}$  is  $\bar{\mathfrak{a}}_j(l+1) + \bar{\mathfrak{m}}^l \otimes_k \mathcal{O}_{Z_{l+1}}$ ,
- (iii)  $\mathfrak{a}_{ij} + \mathfrak{m}^l = \bar{\mathfrak{a}}_j(l)_{s_l(i)} R$  for  $i \in I_l$ , where  $\bar{\mathfrak{a}}_j(l)_z$  is the ideal in  $\bar{R}$  given by  $\bar{\mathfrak{a}}_j(l)$  at  $z \in Z_l$ ,
- (iv)  $s_l(I_l)$  is dense in  $Z_l$ , and
- (v)  $I_{l+1} \subset I_l$  and  $t_{l+1} \circ s_{l+1} = s_l|_{I_{l+1}}$ .

The construction of  $\mathcal{F}$  using Hilbert schemes is exposed in [5, Section 4]. In general, there exist essentially different families of approximations.

For a field extension  $K$  of  $k$ , we set  $\bar{R}_K = \bar{R} \otimes_k K = K[x_1, \dots, x_d]$  with  $\bar{\mathfrak{m}}_K = \bar{\mathfrak{m}}\bar{R}_K$ , and  $\mathbb{A}_K^d = \text{Spec } \bar{R}_K$  with origin  $\bar{P}_K$ . We also set  $R_K = \widehat{\bar{R} \otimes_k K} = K[[x_1, \dots, x_d]]$  with  $\mathfrak{m}_K = \mathfrak{m}R_K$ , and  $X_K = \text{Spec } R_K$  with closed point  $P_K$ .

**Definition A.2.** Suppose that a family  $\mathcal{F}$  of approximations of  $S$  is given as in Definition A.1. For this  $\mathcal{F}$ , take the union  $K = \varinjlim_l K(Z_l)$  of the function fields  $K(Z_l)$  of  $Z_l$  by the inclusions  $t_{l+1}^*: K(Z_l) \hookrightarrow K(Z_{l+1})$ . Then the *generic limit* of  $S$  with respect to  $\mathcal{F}$  is the  $e$ -tuple  $(\mathfrak{a}_1, \dots, \mathfrak{a}_e)$  of ideals in  $R_K$  such that  $\mathfrak{a}_j + \mathfrak{m}_K^l = \bar{\mathfrak{a}}_j(l)_K R_K$  for all  $l \geq l_0$ , where  $\bar{\mathfrak{a}}_j(l)_K$  is the ideal in  $\bar{R}_K$  given by  $\bar{\mathfrak{a}}_j(l)$  at the natural  $K$ -point  $\text{Spec } K \rightarrow Z_l$ .

*Remark A.3.* We have  $\mathfrak{a}_j = \varprojlim_l \bar{\mathfrak{a}}_j(l)_K$ , by  $\bar{\mathfrak{a}}_j(l)_K = \bar{\mathfrak{a}}_j(l+1)_K + \bar{\mathfrak{m}}_K^l$  from (ii) in Definition A.1.

**Definition A.4.** Let  $\mathcal{F} = (Z_l, (\bar{\mathfrak{a}}_j(l))_j, I_l, s_l, t_{l+1})_{l \geq l_0}$  and  $\mathcal{F}' = (Z'_l, (\bar{\mathfrak{a}}'_j(l))_j, I'_l, s'_l, t'_{l+1})_{l \geq l'_0}$  be families of approximations of  $S$ . A *morphism*  $\mathcal{F}' \rightarrow \mathcal{F}$  consists of dominant morphisms  $f_l: Z'_l \rightarrow Z_l$  for  $l \geq l'_0$ , with  $l'_0 \geq l_0$  imposed, such that

- (i)  $t_{l+1} \circ f_{l+1} = f_l \circ t'_{l+1}$ ,
- (ii) the pull-back of  $\bar{\mathfrak{a}}_j(l)$  by  $\text{id}_{\mathbb{A}_k^d} \times f_l$  is  $\bar{\mathfrak{a}}'_j(l)$ , and
- (iii)  $I'_l \subset I_l$  and  $f_l \circ s'_l = s_l|_{I'_l}$ .

An  $\mathcal{F}'$  is called a *subfamily* of  $\mathcal{F}$  if it is equipped with a morphism  $\mathcal{F}' \rightarrow \mathcal{F}$  as above such that all  $f_l$  are open immersions.

We want to compare minimal log discrepancies over  $X$  and  $X_K$ . The comparison of those for approximated ideals is a consequence of the existence of a family of log resolutions on an open subfamily of triplets and Corollary 2.3.

**Lemma A.5** (cf. [15, Proposition 3.2(ii)]). *Notation as above. Let  $(a_1, \dots, a_e)$  be the generic limit of  $S$  with respect to  $\mathcal{F}$ . Then after replacing  $\mathcal{F}$  with a subfamily,*

$$\text{mld}_{P_K}(X_K, \prod_j (a_j + \mathfrak{m}_K^{l_j})^{r_j}) = \text{mld}_{\bar{P}}(\mathbb{A}_K^d, \prod_j \bar{a}_j(l)z^{r_j})$$

for all  $r_1, \dots, r_e > 0$  and all  $z \in Z_l$ .

We utilise a projective morphism which is descended to  $\mathbb{A}_K^d$ .

**Definition A.6.** A projective morphism  $f_K: Y_K \rightarrow X_K$  is said to be *descendible* if there exists a projective morphism  $\bar{f}_K: \bar{Y}_K \rightarrow \mathbb{A}_K^d$  whose base change to  $X_K$  is  $f_K$ .

**Proposition A.7.** *Let  $f_K: Y_K \rightarrow X_K$  be a projective morphism of  $R_K$ -varieties which is isomorphic outside  $P_K$ . Then  $f_K$  is descendible.*

*Proof.* Assuming  $d \geq 1$ ,  $f_K$  is the blow-up along an ideal  $\mathfrak{n}_K$  in  $R_K$  [22, Theorem 8.1.24]. We may assume  $\text{codim}_{X_K} \text{Cosupp } \mathfrak{n}_K \geq 2$ , then  $\text{Cosupp } \mathfrak{n}_K \subset P_K$ , that is,  $\mathfrak{n}_K$  is an  $\mathfrak{m}_K$ -primary ideal. Thus,  $\mathfrak{n}_K$  is the pull-back of the ideal  $\bar{\mathfrak{n}}_K = \mathfrak{n}_K \cap \bar{R}_K$  in  $\bar{R}_K$ . Since blowing-up commutes with flat base change [22, Proposition 8.1.12(c)], the blow-up of  $\mathbb{A}_K^d$  along  $\bar{\mathfrak{n}}_K$  is base-changed to  $f_K$ . q.e.d.

Let  $f_K: Y_K \rightarrow X_K$  be a descendible projective morphism, descended to  $\bar{f}_K: \bar{Y}_K \rightarrow \mathbb{A}_K^d$ . This  $\bar{f}_K$  is defined over  $k(Z_{l'_0})$  for some  $l'_0 \geq l_0$ . For  $l \geq l'_0$ , one can construct inductively a projective morphism  $\bar{f}'_l: \bar{Y}'_l \rightarrow \mathbb{A}_K^d \times_{\text{Spec } k} Z'_l$  with a non-singular open subvariety  $Z'_l$  of  $Z_l$  such that (i)  $\bar{Y}'_l$  is flat over  $Z'_l$ , (ii)  $Z'_{l+1} \subset t_{l+1}^{-1}(Z'_l)$ , and (iii)  $\bar{f}'_{l+1}$  and  $\bar{f}_K$  are the base changes of  $\bar{f}'_l$ , by generic flatness [9, Corollaire IV.11.1.5]. These  $Z'_l$  with  $I'_l = s_l^{-1}(Z'_l(k))$  form a subfamily  $\mathcal{F}'$  of  $\mathcal{F}$ . Replacing  $\mathcal{F}$  with  $\mathcal{F}'$ , we obtain a commutative diagram

$$(15) \quad \begin{array}{ccccc} Y_K & \xrightarrow{\quad} & Y_l & & \\ & \searrow & \downarrow & \searrow & \\ & \bar{Y}_K & \xrightarrow{\quad} & \bar{Y}_l & \\ f_K \downarrow & & \downarrow f_l & & \downarrow \bar{f}_l \\ X_K & \xrightarrow{\quad} & X \times_{\text{Spec } k} Z_l & & \\ & \searrow & \downarrow & \searrow & \\ & \mathbb{A}_K^d & \xrightarrow{\quad} & \mathbb{A}_K^d \times_{\text{Spec } k} Z_l & \end{array}$$

for  $l \geq l_0$  (the  $l_0$  is replaced) such that (i)  $Z_l$  is non-singular, (ii)  $\bar{f}_l$  is projective, (iii)  $\bar{Y}_l$  is flat over  $Z_l$ , and (iv)  $\bar{f}_{l+1}$ ,  $\bar{f}_K$ ,  $f_l$  and  $f_K$  are the base changes of  $\bar{f}_l$ . In general,  $X_K \rightarrow X \times_{\text{Spec } k} Z_l$  is not the base change of  $\mathbb{A}_K^d \rightarrow \mathbb{A}_K^d \times_{\text{Spec } k} Z_l$ .

Whenever an algebraic object over  $X_K$  descendible to  $\mathbb{A}_K^d$  is specified, by taking a subfamily, one can construct (15) so that it comes from a flat family over  $Z_l$ . For example, suppose that  $E_K \in \mathcal{D}_{X_K}$  with centre  $P_K$  is given. It is realised as a divisor on  $Y_K$  equipped with a log resolution  $f_K: Y_K \rightarrow X_K$  of  $(X_K, \mathfrak{m}_K)$ , which is isomorphic outside  $P_K$ . This  $f_K$  is descended to a log resolution  $\bar{f}_K$  by Proposition A.7, and  $\bar{f}_K$  is extended to a family  $\bar{f}_l$  of log resolutions in (15) by generic smoothness. There exists a prime divisor  $\bar{E}_l$  on  $\bar{Y}_l$  which is base-changed to  $E_K$ . By this observation, Lemma A.5 is refined as follows.

**Lemma A.8** (cf. [15, Proposition 3.2(iii)]). *Notation as above. Fix  $r_1, \dots, r_e > 0$  and  $E_K \in \mathcal{D}_{X_K}$  computing  $\text{mld}_{P_K}(X_K, \prod_j \mathfrak{a}_j^{r_j})$ . Then after replacing  $\mathcal{F}$  with a subfamily, there exists a divisor  $\bar{E}_l$  over  $\mathbb{A}_k^d \times_{\text{Spec } k} Z_l$  for any  $l$ , base-changed to  $E_K$ , such that*

$$\begin{aligned} \text{mld}_{P_K}(X_K, \prod_j \mathfrak{a}_j^{r_j}) &= \text{mld}_{\bar{P}}(\mathbb{A}_k^d, \prod_j \bar{\mathfrak{a}}_j(l)_z^{r_j}) = a_{(\bar{E}_l)_z}(\mathbb{A}_k^d, \prod_j \bar{\mathfrak{a}}_j(l)_z^{r_j}), \\ \text{ord}_{E_K} \mathfrak{a}_j &= \text{ord}_{E_K}(\mathfrak{a}_j + \mathfrak{m}_K^l) = \text{ord}_{(\bar{E}_l)_z} \bar{\mathfrak{a}}_j(l)_z < l, \end{aligned}$$

for all  $z \in Z_l$ .

We apply the ideal-adic semi-continuity of log canonicity by Kollár, and de Fernex, Ein and Mustață.

**Theorem A.9** ([19], [5], [6, Proposition 2.20]). *Let  $Q \in Y$  be a germ of an lc variety and set  $\hat{Y} = \text{Spec } \widehat{\mathcal{O}_{Y,Q}}$  with closed point  $\hat{Q}$ . Let  $\mathfrak{a} = \prod_j \mathfrak{a}_j^{r_j}$  be an  $\mathbb{R}$ -ideal on  $\hat{Y}$ . Suppose  $\text{mld}_{\hat{Q}}(\hat{Y}, \mathfrak{a}) = 0$  and it is computed by  $\hat{E} \in \mathcal{D}_{\hat{Y}}$ . If an  $\mathbb{R}$ -ideal  $\mathfrak{b} = \prod_j \mathfrak{b}_j^{r_j}$  on  $\hat{Y}$  satisfies  $\mathfrak{a}_j + \mathfrak{p}_j = \mathfrak{b}_j + \mathfrak{p}_j$  for all  $j$ , where  $\mathfrak{p}_j = \{u \in \mathcal{O}_{\hat{Y}} \mid \text{ord}_{\hat{E}} u > \text{ord}_{\hat{E}} \mathfrak{a}_j\}$ , then  $\text{mld}_{\hat{Q}}(\hat{Y}, \mathfrak{b}) = 0$ .*

**Corollary A.10.** *In Lemma A.8, if  $\text{mld}_{P_K}(X_K, \prod_j \mathfrak{a}_j^{r_j}) = 0$ , then  $\text{mld}_P(X, \prod_j \mathfrak{a}_{ij}^{r_j}) = 0$  for any  $i \in I_l$  on a subfamily. In particular, if  $(X_K, \prod_j \mathfrak{a}_j^{r_j})$  is lc, then so is  $(X, \prod_j \mathfrak{a}_{ij}^{r_j})$ .*

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